MATHEMATICAL WORLD VOLUME 30

The Mathematics of Voting and Elections: A Hands-On Approach

SECOND EDITION

Jonathan K. Hodge
Richard E. Klima

AMS American Mathematical Society

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Preface

When we wrote the first edition of this book back in 2004, topics from the social sciences had been gradually making their way into mathematics textbooks for about a decade. These topics remain popular nearly 15 years later, and a renewed interest in politics—sparked largely by the advent of social media and recent developments in both the United States and abroad—has provided a natural opportunity to revise, update, and expand on the material from the first edition.

At its core, this book remains a resource that can be used for independent study, as part of a liberal arts mathematics course, or for a stand-alone course on the mathematics of voting and elections. We and other users have had success in all three of these contexts.

Pedagogically, this book was inspired by our involvement with the Legacy of R. L. Moore Project and our ongoing use of inquiry-based learning in our own teaching. We aimed to capture the spirit of a Moore-style, inquiry-based approach, but we also wanted to make sure that the resulting text was accessible to a non-mathematical audience. To do so, we made a point of writing in a casual and non-threatening tone. We also tried to place each topic of study in its appropriate historical context—and to tell an interesting and engaging story through our investigations.

If you are accustomed to working with more traditional mathematics texts, you may notice several common features that are missing from this one. For one thing, we have not included any worked-out examples within the body of the text. Instead, we have provided "starred" questions whose answers appear in full or in part at the end of each chapter. These questions are intended to help the reader gauge their own understanding of foundational definitions and concepts before moving on to more difficult material. Thus, our starred questions play the same role as examples in other texts, but they do so in a way that forces the reader to more actively engage with the ideas being developed.

We have also not included any repetitious or skill-and-drill type problems, but have instead focused on asking questions that require in-depth analysis and critical thinking skills. In fact, we use these questions not only

to supplement material presented in the text, but also as an essential part of the development of the book's key concepts. For this reason, **it is important for readers to approach this book with pencil and paper close at hand, and to carefully work through each question within the main body of the text before moving on.** The only exceptions to this rule are the Questions for Further Study provided at the end of each chapter, which can be completed more selectively.

It would be difficult to cover all of the material in this book in a one-semester course on voting theory. Certain sections and chapters, however, can be omitted without loss of continuity. Specifically:

- Chapters 1–4 introduce the basics of mathematical voting theory up to Arrow's Theorem, and they should be covered in order. However, the proof of May's Theorem (beginning on page 7) can be omitted from Chapter 1 without causing any difficulties later on.
- Chapter 5 walks the reader through a proof of Arrow's Theorem and then discusses some options for resolving the difficulties revealed by the theorem. This entire chapter can be omitted, although it would be worthwhile to at least cover the section on approval voting (beginning on page 80).
- Chapter 6, which is new to the second edition, covers manipulation of voting systems, focusing mainly on a proof of the Gibbard-Satterthwaite Theorem. The content of this chapter is challenging at times and will appeal to more mathematically sophisticated readers. The entire chapter can be omitted, although the first few pages provide an accessible overview of strategic voting that is worth exploring if time permits.
- Chapters 7 and 8 go together and should be covered in order. They rely only casually on the material in Chapters 1–4.
- Chapters 9, 10, and 11 are essentially independent from the rest of the text and from each other; they can be covered in any order, or omitted. Chapter 9 uses some terminology from Chapters 7 and 8 (specifically, the language of coalitions and power indices) but requires only a surface-level understanding of these ideas.
- Chapter 12—also new to the second edition—covers a host of topics related to gerrymandering. Other than a few passing references, mainly to Chapter 11 (on apportionment), it does not rely heavily on the content of the preceding chapters.

Although our own approach to teaching with this book involves group work, student presentations, discussions, debates, and virtually no lecturing whatsoever, we would encourage instructors to experiment with other techniques and class formats as well. We hope that this book serves as a useful starting point for whatever your goals might be, and we look forward to hearing from you if you have any questions, comments, or suggestions.

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When we first wrote this book, during the summer of 2004, we had the pleasure of working with three outstanding student assistants: Mike Cheyne, Pete Schwallier, and Dave Wils. Their insights and perspectives were invaluable. Since the publication of the first edition, we have also received feedback from numerous users of the book, much of which we have incorporated into this second edition.

Finally, the new chapter on gerrymandering (Chapter 12) was inspired in large part by the October 2017 Geometry of Redistricting workshop held at the University of Wisconsin–Madison. We're grateful to the workshop organizers—Moon Duchin, Mira Bernstein, and Jordan Ellenberg—not only for their work in putting together an outstanding program, but also for raising awareness of these important issues and illuminating the role that mathematicians can play in addressing them.

Chapter 1

What's So Good About Majority Rule?

Whenever you find yourself on the side of the majority, it is time to reform.
– Mark Twain

Focus Questions

In this chapter, we'll explore the following questions:

- What are some of the systems that can be used to decide the winner of an election with two candidates? What are the strengths and weaknesses of these systems?
- What are some of the criteria that can be used to evaluate voting systems for two-candidate elections?
- What is special or unique about majority rule? Which theorem establishes this uniqueness?
- What is a quota system? How are quota systems related to majority rule and its unique features?

The Mayor of Stickeyville

Warmup 1.1. The time has come for the citizens of Stickeyville to elect a new mayor. Two candidates are running for the office: Mike Dowell and Laura Stutzman. What method should be used to decide the winner?

Did that warmup question seem too easy? Did its answer seem too obvious? If so, perhaps you should consider my proposal for deciding the winner of the election: I have a friend named Stan who lives in Stickeyville. I propose that, to decide the election, the citizens of Stickeyville should all vote; that's only fair. However, after the voting is done, I think that whoever

Stan votes for should be declared the winner, regardless of how any of the other voters vote.

Question 1.2.* Suppose all 101 of the citizens of Stickeyville show up at the polls to vote on election day. If 100 of them vote for Dowell and Stan votes for Stutzman (his girlfriend), who would win the election under the method described in the preceding paragraph?

Your answer to the previous question probably convinced you that the method I proposed for deciding the winner of the Stickeyville mayoral election isn't all that fair. It is, after all, equivalent to a **dictatorship**, which—by definition—is not very democratic. As you might have observed, the main fault of a dictatorship is that it does not treat all of the voters equally. Wouldn't you agree that my method treats Stan (the dictator) in a rather special way?

Allow me to propose another option: Dowell wins, no matter how anyone (including Stan) votes.

Question 1.3.* Does the "Dowell wins" method treat all of the voters equally? Explain.

In spite of your answer to the last question, you probably don't believe that my new proposal is any better than my old one. In fact, the method I suggested—of declaring Dowell the winner no matter how anyone votes—has the decidedly undemocratic name of **imposed rule**. In imposed rule, the outcome is ultimately decided before the election even takes place. Unlike a dictatorship, where at least the dictator's vote matters, with imposed rule nobody's vote matters. Since the winner is decided beforehand, imposed rule suffers from a different problem than a dictatorship: It doesn't treat all of the *candidates* equally. It would be quite an understatement to say that Dowell is favored in the election under imposed rule; in truth, it would be impossible for Dowell to lose, even if everyone voted for Stutzman.

Let me make just one more proposal for deciding the winner of the Stickeyville mayoral election: Each voter should vote for the candidate they want to win the election. The votes for each candidate should then be

* Starred questions are intended to help you gauge your understanding of fundamental concepts and definitions before moving on to more difficult material. Partial or complete answers to these questions are provided at the end of each chapter, and we encourage you to use these answers to check your work. However, you should resist the temptation to look ahead to an answer until after you have made a serious attempt to answer the question yourself. You should check the answer only after you are convinced that you have come up with a correct solution on your own. Then if your answer does not agree with the answer at the end of the chapter, you should try to discover what errors you may have made and correct these errors before moving on to subsequent questions. You may want to consider talking to someone else if you need help figuring out how to correct or improve your solutions. It is also important to note that in the answers provided at the end of the chapters, we often leave out a number of details that you should include when you write your own solutions.

counted, with the candidate having the *smallest* number of votes declared the winner. Not surprisingly, this method is called **minority rule**.

Question 1.4.* Suppose again that all 101 citizens of Stickeyville show up at the polls, with 100 voting for Dowell and one (Stan) for Stutzman. Who would win the election under minority rule?

Question 1.5. Suppose that Stan convinces 50 of the 100 Dowell voters to change their minds and vote for Stutzman. Who would win the election in this case under minority rule?

Question 1.6. Does minority rule treat all of the voters equally? Does it treat all of the candidates equally? Explain.

Question 1.7. Under minority rule, is it beneficial or detrimental to a candidate to receive additional votes? Explain.

Anonymity, Neutrality, and Monotonicity

So far, we have considered three different methods for deciding the winner of the Stickeyville mayoral election. Such methods are typically called **voting systems**, although it's important to note that this terminology refers not only to the way votes are cast in a particular election, but also to the way in which the winner of the election is determined from the individual votes.

One way to evaluate the fairness of a given voting system is to identify certain desirable properties that we think the system should satisfy, and then see if it actually does satisfy these properties. Recall that we did this for each of the three voting systems we considered in the previous section. In fact, the properties we identified are well known; they have precise names and definitions, which we'll now state.

Definition 1.8.

- A voting system for a two-candidate election is **anonymous** if it treats all of the voters equally, meaning that if any two voters traded votes, the outcome of the election would remain the same.
- A voting system for a two-candidate election is **neutral** if it treats both of the candidates equally, meaning that if *every* voter switched their vote from one candidate to the other, the outcome of the election would change accordingly—the winning candidate would become the losing candidate and the losing candidate would become the winning candidate. (And in the case of a tie, everyone switching their vote would have no effect on the outcome of the election.)
- A voting system for a two-candidate election is **monotone** if it is impossible for a winning candidate to become a losing candidate by gaining votes (and not losing any others) or for a losing candidate

to become a winning candidate by losing votes (and not gaining any others).

Question 1.9.* Suppose three children, Zoey, Asher, and Caden, are trying to decide which of their parents, Juan or Melisa, should plan their next family vacation. To make this decision, they decide to hold an election using a voting system invented by their friend Clark (who, incidentally, went to Walt Disney World on his family vacation last year). Three possible combinations of votes by Zoey, Asher, and Caden, and the outcome that Clark's voting system would produce for each combination, are shown in Table 1.1. In the table, J represents a vote for Juan and M a vote for Melisa.

Zoey	Asher	Caden	Winner
J	M	M	J
J	J	M	M
M	M	J	M

TABLE 1.1. Results of Clark's voting system

(a) Which of the three properties described in Definition 1.8 are satisfied by Clark's voting system? Explain.

(b) Is Clark's voting system equivalent to any of the other three voting systems that we've investigated? Why or why not?

Question 1.10.* Suppose you wanted to give an example to convince a friend that a particular voting system is *not* anonymous. According to Definition 1.8, what features would your example need to have?

Question 1.11. Use your answer to Question 1.10 to explain clearly why a dictatorship is not anonymous.

Question 1.12. Explain clearly why a dictatorship is both neutral and monotone.

Question 1.13. Which of the three properties described in Definition 1.8 are satisfied by imposed rule? Which are not satisfied? Give a convincing argument to justify each of your answers.

Question 1.14. Which of the three properties described in Definition 1.8 are satisfied by minority rule? Which are not satisfied? Give a convincing argument to justify each of your answers.

Question 1.15.* Summarize the properties of the three voting systems we've looked at so far by completing the following table.

	Anonymous	Neutral	Monotone
Dictatorship	No	Yes	Yes
Imposed Rule			
Minority Rule			

Majority Rule and May's Theorem

By now, you may feel like we've been dancing around the main point for quite some time. In a sense, you're right. We haven't yet found the perfect voting system for the Stickeyville mayoral election, but we also haven't considered the most obvious choice.

Unless you were trying to be unique, your answer to Warmup 1.1 was probably something like this:

> *Each voter should vote for the candidate they want to win the election. The votes for each candidate should then be counted, and the candidate with the largest number of votes should be declared the winner.*

If you were particularly clever, you might have added something about how ties should be broken. For now, we won't worry about this. We'll just assume that if a tie occurs, it will be broken by some separate procedure.

For two-candidate elections, the voting system described in the preceding paragraph is known as **majority rule**. Majority rule seems on the surface to be much more reasonable than any of the other three systems we've considered. But how does it stand up to the three desirable properties we've discussed—anonymity, neutrality, and monotonicity? As it turns out, majority rule satisfies all three!

Question 1.16. Write a clear and precise explanation of why majority rule is anonymous, neutral, and monotone.

So we've finally found a voting system that behaves nicely, at least according to some reasonably defined standards. The next natural question to ask is this: Are there any other voting systems for two-candidate elections (besides majority rule) that also live up to these standards? The answer to this question comes from mathematician Kenneth May, who proved the following theorem in a paper from 1952 [**35**]:

May's Theorem. *In a two-candidate election with an odd number of voters, majority rule is the only voting system that is anonymous, neutral, and monotone and avoids the possibility of a tie.*

Question 1.17. In a two-candidate election, why is it especially important for a voting system to avoid the possibility of a tie?

May's Theorem is actually a fairly simple consequence of another theorem regarding *quota systems*, which we'll consider next.

Quota Systems

Definition 1.18. A voting system is called a **quota system** if there is some number q, called the **quota**, such that a candidate will be declared a winner of an election if and only if they receive at least q votes.

A few observations are in order here:

- The phrase "if and only if" has a special meaning in mathematics. In the above definition, it means that
  - if a candidate receives at least q votes, then they will be declared a winner; *and*
  - if a candidate does not receive at least q votes, then they will not be declared a winner.
- In a quota system with two candidates, there could end up being two winners (if both candidates reach the quota) or two losers (if neither candidate reaches the quota). In either case, a separate procedure would have to be used to break the tie.
- In a quota system, the quota can depend on the number of voters casting ballots. For instance, in the state of California, any proposal by a local government to assess a new tax or fee must be approved by two thirds of the electorate. So if 900 residents voted on such a proposal, then 600 "yes" votes would be required for the measure to pass; thus, the quota for the system would be 600. But if 900,000 residents voted, then the quota for the system would be 600,000 instead of 600.

Question 1.19.* Suppose the citizens of Stickeyville decide to use a quota system to elect their new mayor. What would the outcome of the election be in each of the following scenarios?

(a) Quota = 51; Dowell receives 51 votes and Stutzman 50.

(b) Quota = 40; Dowell receives 51 votes and Stutzman 50.

(c) Quota = 60; Dowell receives 51 votes and Stutzman 50.

(d) Quota = 101; Dowell receives 100 votes and Stutzman 1.

(e) Quota = 0; no additional information.

Question 1.20.* Is Clark's voting system from Question 1.9 a quota system? Why or why not?

Question 1.21. Which of the four voting systems that we've studied (dictatorship, imposed rule, minority rule, and majority rule) are quota systems? Give a convincing argument to justify your answer for each system.

We're now ready to investigate the following theorem, of which May's Theorem is a consequence.

Theorem 1.22. *If a voting system for an election with two candidates is anonymous, neutral, and monotone, then it is a quota system.*

Proving a claim such as Theorem 1.22 is kind of like solving a mystery. Our suspect, an unknown voting system that we'll call V, has left behind a trail of clues that will eventually allow us to conclude beyond any reasonable doubt that V is actually a quota system. The first three clues are that V is anonymous, neutral, and monotone. We also know that for any combination of votes in a two-candidate election, V must be able to tell us exactly which candidate (or candidates) it would choose as the winner. (This is, after all, what voting systems do.) Keeping all of this in mind, what we really need to do is ask V the right questions. The answers to these questions will help us identify a value that *could* work as a quota for V. Once we've found this potential quota, say q, our last step will be to argue that V is not just any voting system, but is in fact a quota system with a quota of exactly q.

The next question suggests both the kind of information that we might want to try to extract from V and how we might use this information to determine a potential quota for V.

Question 1.23. Suppose that for an election with two candidates, Jen and Brian, you know the following about V. (Assume that Joel and Grace are just two of many voters in the election.)

- If nobody votes for Jen, then V *will not* choose Jen as a winner.
- If only Joel votes for Jen, then V *will not* choose Jen as a winner.
- If Joel and Grace vote for Jen, then V *will* choose Jen as a winner.

Using only this information and the fact that V is anonymous, neutral, and monotone, could you then say that V is a quota system? If so, what would the quota be? Give a convincing argument to justify your answer, being sure to specify exactly where in your argument you use each of the properties of anonymity, neutrality, and monotonicity. (Hint: You may want to go back and carefully re-read Definition 1.18.)

Question 1.23 demonstrates that once we have extracted the right information, V begins to look a lot like a quota system. Of course, in that question, the information we needed was basically handed to us on a silver platter. We can't expect to always be this lucky, but as we noted earlier, we should be able to find out all the information we need by simply asking V the right questions.

Question 1.24.* Consider an election with two candidates, A and B, and n voters, whom we'll label v_1, v_2, v_3, ..., v_n. (Note that n just represents

some arbitrary number of voters.) Suppose that we ask V the following sequence of questions regarding the election:

- If nobody votes for candidate A, would you choose A as a winner?
- If only v_1 votes for candidate A, would you choose A as a winner?
- If v_1 and v_2 vote for candidate A, would you choose A as a winner?
- If v_1, v_2, and v_3 vote for candidate A, would you choose A as a winner?

  $\vdots$

- If v_1, v_2, v_3, $\ldots$, v_{n-1}, and v_n all vote for candidate A, would you choose A as a winner?

Explain how V's answers to these questions could be used to identify a value that might work as a quota for V. Would it ever be possible to identify this potential quota without asking *all* of these questions? Why or why not?

Question 1.25.* Suppose that the method suggested in Question 1.24 was used to identify a potential quota, say q, for V. Clearly explain why each of the following statements would have to be true. Your answers should make use of the fact that V is anonymous, neutral, and monotone.

(a) If exactly q voters (no matter which ones they are) vote for candidate A, then V would choose A as a winner.

(b) If more than q voters (no matter which ones they are) vote for candidate A, then V would choose A as a winner.

(c) If exactly $q-1$ voters (no matter which ones they are) vote for candidate A, then V would not choose A as a winner.

(d) If fewer than $q-1$ voters (no matter which ones they are) vote for candidate A, then V would not choose A as a winner.

(e) All of the above conclusions also apply to candidate B.

Question 1.26. Use your answers to Questions 1.24 and 1.25 to clearly explain why Theorem 1.22 is true. That is, explain why in a two-candidate election, every voting system that is anonymous, neutral, and monotone must be a quota system.

Back to May's Theorem

Now that we understand Theorem 1.22 and why it is true, we can finally begin to see why May's Theorem is true. Recall that Theorem 1.22 tells us that in an election with two candidates, every voting system that is anonymous, neutral, and monotone must be a quota system.[1] May's Theorem tells us that if we assume in addition that the number of voters in the election is

[11] Incidentally, it's not too hard to see that the backwards version of this statement is also true; that is, every quota system is in fact anonymous, neutral, and monotone.

odd and that ties are not allowed, then not only must the system be a quota system, but it must be identical to majority rule. Thus, we can prove May's Theorem by arguing that for a two-candidate election with an odd number of voters, majority rule is the only quota system that avoids ties.

Question 1.27.* Suppose majority rule (a quota system) is used in a two-candidate election with n voters. Describe how to find the quota in this case. (Hint: You know an awful lot about majority rule, so use your intuition, remembering that the quota must be a whole number.)

We'll now argue that for an odd number of voters, the only quota system that avoids the possibility of ties is the one with the quota that you just found in Question 1.27.

Question 1.28.* Suppose that a two-candidate election is to be decided using a quota system with quota q. Let a and b denote the number of votes received by the two candidates, A and B, respectively.

(a) How must a and b compare to q for the election to result in a tie?

(b) How must a and b compare to q for the election to *not* result in a tie?

Question 1.29. Suppose that a two-candidate election with n voters is to be decided using a quota system with quota q.

(a) Suppose q is greater than the quota you found for majority rule in Question 1.27. Give an example to show that, in this case, the outcome of the election could be a tie.

(b) Repeat part (a), but this time assume that q is less than the quota you found for majority rule in Question 1.27.

(c) Suppose n is even and q is exactly equal to the quota you found for majority rule in Question 1.27. Give an example to show that, in this case, the outcome of the election could be a tie.

(d) Suppose n is odd and q is exactly equal to the quota you found for majority rule in Question 1.27. Explain why, in this case, the election *could not* result in a tie.

Question 1.30. Summarize what you learned in this section by writing a clear explanation of how Theorem 1.22 implies May's Theorem. That is, explain why May's Theorem follows from Theorem 1.22.

Question 1.31.

(a) Does there exist a quota system for an election with two candidates that avoids the possibility of ties when there are an even number of voters?

(b) Explain why we need to assume in the statement of May's Theorem that the number of voters is odd.

Questions for Further Study

Question 1.32. In this chapter, we discussed several desirable properties that we might want voting systems to satisfy. Think of another desirable property that we didn't discuss, and explain why you think it would be desirable for voting systems to satisfy this property.

Question 1.33. After delivering a particularly lengthy sermon, the pastor of Long Winds Church is facing a recall vote from his congregation. If two thirds of the voters vote in favor of the recall, then the pastor will become the janitor and the janitor will become the pastor. Otherwise, the pastor will continue to preach and the janitor will continue to mop.

(a) Explain how the procedure for deciding this election could be viewed as a quota system according to the definitions in this chapter.

(b) Suppose that a couple in the church, Greg and Gail, inadvertently cast opposing votes, with Greg voting in favor of the recall and Gail against it. Upon realizing this, Greg remarks to Gail, "Well, I guess our votes cancelled each other out!" Is Greg correct, or would it have been possible for his and Gail's opposing votes to have changed the outcome of the election? Give a convincing argument or example to justify your answer.

Question 1.34. Write a short biography of Kenneth May, including his most important contributions both inside and outside of voting theory.

Question 1.35. If there were only two candidates running for president of the United States, would majority rule dictate the winner of the election? Explain.

Question 1.36. If there were only two candidates running for president of the United States, would majority rule dictate the winner of Michigan's electoral votes? What about Nebraska's? Explain.

Question 1.37. Investigate the electoral and popular vote totals received by the candidates in the 1876 U.S. presidential election. Write a summary of your findings, and explain how they relate to our study of majority rule.

Question 1.38.

(a) If the United States Congress voted in an attempt to override a presidential veto, would majority rule dictate whether the override was successful? Explain.

(b) If the United States Congress voted in an attempt to override a presidential veto, would a quota system dictate whether the override was successful? If so, what would the quota for the system be?

Question 1.39. Find a magazine, newspaper, or web site that describes an election with exactly two candidates in which majority rule dictated the winner of the election. Write a detailed summary of your findings.

Question 1.40. Find a magazine, newspaper, or web site that describes an election with exactly two candidates in which majority rule *did not* dictate the winner of the election. Write a detailed summary of your findings.

Question 1.41. Research the pope selection process in the Roman Catholic church. Who are the candidates? Who are the voters? Is the pope selected by majority rule? Write a detailed summary of your findings.

Question 1.42. Consider an election with two candidates, Mya and Luis, and three voters, Ahmed, Beatriz, and Clara. Suppose that if Ahmed and Beatriz vote for Mya, and Clara votes for Luis, then Mya will win. Suppose also that the voting system being used is anonymous, neutral, and monotone. Using only this information, determine what the outcome of the election would be for each of the seven other combinations of votes. Clearly explain your reasoning, including where you used each of the properties of anonymity, neutrality, and monotonicity.

Question 1.43. Consider a voting system for an election with two candidates in which voters from each of two separate parties—say Republican and Democratic—vote separately, and a candidate is declared a winner if and only if they receive more than half of the votes from *each* party. (So, for example, a candidate who received 60% of the Republican votes but only 40% of the Democratic votes would not be declared a winner.)

(a) Is this system anonymous? Give a convincing argument or example to justify your answer.

(b) Is this system neutral? Give a convincing argument or example to justify your answer.

(c) Is this system monotone? Give a convincing argument or example to justify your answer.

Question 1.44. Repeat Question 1.43, but this time assume that a candidate is declared a winner if and only if they receive votes from *more than half* of the Republicans and *less than half* of the Democrats.

Question 1.45. Suppose that in an election with two candidates, a candidate is declared a winner if and only if they receive an even number of votes. Decide whether such a system is anonymous, neutral, and monotone. Give a convincing argument or example to justify your answer for each property.

Question 1.46. Research the tie-breaking methods used in various U.S. states for general elections. In which state is it possible for the winner to be decided by a game of poker? In which state was a recent election decided by drawing a name from a bowl?

Question 1.47. A devious politician has hired you to find or invent a voting system that violates all three of the properties of anonymity, neutrality, and monotonicity. Does such a voting system exist? If so, describe one such system. If not, explain why no such system can exist.

Answers to Starred Questions

1.2. Since Stan's vote is the only one that matters, Stutzman would win.

1.3. The "Dowell wins" method treats all of the voters equally since nobody's vote matters.

1.4. Stutzman would win since she received fewer votes than Dowell.

1.9. (a) None of the three properties are satisfied. The first two rows of Table 1.1 show that Clark's system is not monotone, the first and third rows show that it is not anonymous, and the last two rows show that it is not neutral.

(b) Clark's system is not equivalent to a dictatorship, since neither Zoey, Asher, nor Caden always agrees with the winning outcome. It is not equivalent to imposed rule, since the winner is not the same for each combination of votes. And it is not equivalent to minority rule, since in the third row Juan receives fewer votes than Melisa but still loses.

1.10. Your example would need to exhibit two combinations of votes that are the same except for two voters having swapped ballots—and yet the two combinations produce different winners.

1.15. The table can be completed as follows.

	Anonymous	Neutral	Monotone
Dictatorship	No	Yes	Yes
Imposed Rule	Yes	No	Yes
Minority Rule	Yes	Yes	No

1.19. (a) Dowell would win and Stutzman would lose since only Dowell receives at least as many votes as the quota.

(b) Dowell and Stutzman would both win since they both receive at least as many votes as the quota.

(c) Dowell and Stutzman would both lose since they both receive fewer votes than the quota.

(d) Dowell and Stutzman would both lose since they both receive fewer votes than the quota.

(e) Dowell and Stutzman would both win since they would both have to receive at least as many votes as the quota.

1.20. Clark's voting system cannot be a quota system. There are only four possibilities for the quota: 0, 1, 2, and 3. If the quota were 0 or 1, then according to Table 1.1, all three combinations of votes would result in a tie. If the quota were 2, then Melisa would win and Juan would lose in the first row, and Juan would win and Melisa would lose in the second row. And if the quota were 3, then no combination of votes would produce a winner.

1.24. The first question to which V answered yes would indicate the quota for the system. For instance, if V answered yes to the first question, then the quota would be 0, and the system would always result in a tie with both candidates chosen as winners. But if V answered no to the first question but yes to the second, then the quota would be 1. If V answered no to all of the questions, then the quota could be any number greater than n, and the system would always result in a tie with no winner.

1.25. Your answers to parts (a) and (c) should use anonymity, your answers to parts (b) and (d) should use monotonicity, and your answer to part (e) should use neutrality.

1.27. If n is even, then the quota for majority rule is $\frac{n}{2} + 1$. If n is odd, then the quota for majority rule is $\frac{n}{2} + \frac{1}{2}$.

1.28. (a) For the election to result in a tie, a and b would have to both be greater than or equal to q, or both be less than q.

(b) For the election to not result in a tie, exactly one of a and b would have to be greater than or equal to q and the other would have to be less than q.

Chapter 2

Le Pen, Nader, and Other Inconveniences

Focus Questions

In this chapter, we'll explore the following questions:

- What is the plurality method for determining the winner of an election? How does it differ from majority rule?
- What is the Borda count? How is it defined, and where is it commonly used?
- What is the majority criterion? Is it satisfied by plurality? By the Borda count?
- How do the plurality and Borda count methods relate to May's Theorem?

Warmup 2.1. The popular vote totals from the state of New Hampshire in the 2016 U.S. presidential election are given in Table 2.1.

(a) In this election, did any of the candidates receive a *majority* (i.e., more than half) of the popular votes cast in the state of New Hampshire?

(b) If Hillary Clinton and Donald Trump had been the only candidates in the 2016 U.S. presidential election, do you think that Trump might have possibly received more popular votes than Clinton in New Hampshire?

As you may recall, the 2016 U.S. presidential election was hotly contested, and the wounds from a bruising campaign season were not quick to heal. Many battleground states, like New Hampshire, had remarkably close results. As a percentage of the votes cast in New Hampshire, the difference

Candidate	Popular Votes
Hillary Clinton	348,526
Donald Trump	345,790
Gary Johnson	30,777
Others	19,203

TABLE 2.1. 2016 U.S. presidential election in New Hampshire

between the vote totals of the two main candidates was a minuscule 0.368%. Michigan, which was won by Trump, was even closer at 0.223%.

Both of these differences, however, pale in comparison to the microscopic 0.009% difference between the vote totals in Florida for the two main candidates in the 2000 U.S. presidential election (shown in Table 2.2).[1]

Candidate	Popular Votes
George W. Bush	2,912,790
Al Gore	2,912,253
Ralph Nader	97,488
Others	40,579

TABLE 2.2. 2000 U.S. presidential election in Florida

To make matters worse, counting and recounting of the popular votes in Florida delayed final certification of the results until over a month after the ballots were cast. And unlike in 2016, when no single competitive state alone swung the outcome of the national election, Florida was the deciding factor in the 2000 election. In other words, the winner in Florida would have won the national election, whether it had been George W. Bush or Al Gore. Ultimately, Bush won the state by a razor-thin margin, resulting in him winning the national election and becoming the 43rd president of the United States.

Many political scientists speculate that if Ralph Nader had not been a candidate, Gore would have won Florida and, consequently, the national election. In other words, Nader was a *spoiler candidate* in the sense that he did not stand a realistic chance of winning the national election, but he may have affected its outcome nonetheless.

[11] It's important to note here that U.S. presidential elections are not decided by simply declaring the winner to be the candidate who receives the most popular votes, but rather by a system known as the *Electoral College*, which we will study in Chapter 9. In fact, Trump was the overall winner of the 2016 election even though Clinton received more popular votes than Trump nationwide—65,853,516 for Clinton to 62,984,825 for Trump. This behavior was also exhibited in the 2000 election, where the overall winner, Bush, received only 50,456,002 popular votes nationwide, while Gore received 50,999,897.

Such problems are not inherent to only U.S. elections, of course. A similar situation occurred in the 2002 French presidential election, when in the initial round of voting, political outsider Jean-Marie Le Pen finished second, beating mainstream candidate Lionel Jospin by 0.683% of the votes cast. This prevented an anticipated very close head-to-head final contest between political rivals Jacques Chirac and Jospin. Instead, the final contest was an overwhelming victory for Chirac over Le Pen. Both of these examples serve to illustrate how third-party candidates have the potential to introduce complexities into elections that are not present when only two candidates are involved. In this chapter, we'll begin to investigate these complexities and some ways of dealing with them.

The Plurality Method

As we've seen, Hillary Clinton did not receive a majority of the votes cast in New Hampshire in the 2016 U.S. presidential election. She did, however, receive more votes than any of the other candidates. To state this situation more concisely, we could say that Clinton received a **plurality** of the votes cast in New Hampshire.

Question 2.2.*

(a) For elections with two candidates, explain why the words *plurality* and *majority* mean exactly the same thing.

(b) For elections with more than two candidates, explain why the words *plurality* and *majority* do not mean exactly the same thing.

As you saw in Question 2.2, the kind of elections we studied in Chapter 1 (those involving only two candidates) gave us no reason to distinguish between a plurality winner and a majority winner. In fact, we actually used the definition of plurality (without using the word itself) when we defined majority rule in Chapter 1.

Warmup 2.1 demonstrates that the situation is more complicated for elections with more than two candidates. Since it is possible for a candidate to win an election without receiving a majority of the votes cast, we must make a distinction between receiving the *largest* number of votes and receiving *more than half* of the votes. To do so, we'll adopt the following definitions.

Definition 2.3. Consider an election with more than two candidates.

- **Majority rule** is the voting system that elects the candidate who receives more than half of the votes, if such a candidate exists. If no candidate receives more than half of the votes, then majority rule results in a tie, with no declared winner.
- The **plurality** method (or just *plurality* for short) is the voting system that elects the candidate who receives the largest number of votes,

even if it is less than half of the total number of votes cast. Plurality results in a tie, with multiple winners elected, when two or more candidates receive exactly the same number of votes and this number is more than the number of votes received by any of the other candidates.

Question 2.4.*

(a) Which of the two methods described in Definition 2.3 is more likely to result in a tie?

(b) If a candidate wins an election under majority rule, would that candidate also be guaranteed of winning under plurality?

(c) If a candidate wins an election under plurality, would that candidate also be guaranteed of winning under majority rule?

Question 2.5. Twenty-one people filed paperwork with the U.S. Federal Election Commission as candidates for the 2016 Republican nomination for president. The total number of votes cast in 2016 Republican primaries nationwide was 31,183,841.

(a) Donald Trump received 14,015,993 of these votes. Did he receive a majority of the votes cast in the Republican primaries?

(b) If the winner of the 2016 Republican nomination had been chosen by plurality from these 21 candidates, what is the smallest number of votes Trump could have received and still have had a chance of winning the nomination? (Assume that the number of voters remains fixed at 31,183,841.)

(c) Under the same assumptions as in part (b), what is the maximum number of voters who could have preferred Trump the *least* among the 21 candidates in order for him to still have had a chance of winning the nomination?

(d) Using your answers to parts (b) and (c), formulate a well-written criticism of the plurality method. You don't have to agree with your argument, but put yourself in the shoes of a critic and try to predict the type of argument that might be made against plurality.

The Borda Count

Although a number of important elections are decided by the plurality method, it is only one of several voting systems that we'll investigate. For instance, many collegiate sports polls use a version of a method known as the **Borda count** to form their rankings. The Borda count, which uses

a point system to determine overall rankings, is named in honor of Jean-Charles de Borda, a French astronomer, mathematician, military officer, and voting theory pioneer.

Question 2.6.* Part of the 1971 Associated Press preseason college football poll, which was formed using a version of the Borda count, is shown in Table 2.3.

(a) Which team would have been ranked first if the ranking had been based solely on the number of first-place votes received?

(b) Did the team from part (a) receive a *majority* of the first-place votes?

Rank	Team	Points	First-Place Votes
1	Notre Dame	885	15
2	Nebraska	870	26
3	Texas	662	5
4	Michigan	593	1
5	Southern California	525	1
6	Auburn	434	1
⋮	⋮	⋮	(all 0)
20	Northwestern	58	1
⋮	⋮	⋮	(all 0)

TABLE 2.3. AP college football poll, 1971 preseason

We haven't learned much about the Borda count yet, but we can make one very important observation from Question 2.6: Under the Borda count, it is possible for a candidate (or team, as it is in this case) to be viewed as the most desirable by a majority of the voters in an election, but still not win! When a voting system is capable of behaving this badly, we say that the system violates the **majority criterion**. (We'll define this term more precisely in a bit.) Incidentally, it's worth noting that plenty of other anomalies can occur with the Borda count. For instance, in the 1994 Associated Press preseason college football poll, Nebraska received a plurality of the first-place votes (though not a majority), but was ranked *fourth*!

At this point, you might be wondering what kind of strange and perverse system would fail to elect a candidate who is the most desirable choice of more than half of the voters in an election. Would it surprise you to learn that many political scientists and mathematicians have suggested that the Borda count is not strange or perverse at all, but is in fact superior to plurality? We'll investigate this claim more fully later on, but first we need to take care of a few preliminary details that were not around to bother us when we were considering elections with only two candidates.

Preference Orders

Notice that in a two-candidate election, a voter's ranking of the candidates from most to least desirable is determined entirely by their first choice. For instance, in the Stickeyville mayoral election from Chapter 1, if I vote for Stutzman, you know that Stutzman is my first choice and Dowell is my second choice. Because there are only two candidates, once you know who I voted for, then you know everything that could possibly be known about my preferences for all of the candidates.

Suppose, however, that I told you I had voted for Hillary Clinton in the 2016 U.S. presidential election. Would you know who my second and third choices were? You might be able to make an educated guess, but without more information, you couldn't know for sure. In order for you to fully understand my preferences, I would have to provide you with some kind of ranking, or what we'll call a **preference order** (sometimes also called a *preference ballot* or *preference list*). If we focus only on the three most popular candidates (Hillary Clinton, Donald Trump, and Gary Johnson) and ignore the others, my preference order would have had to have been one of the following:

Rank	Candidate
1	Clinton
2	Trump
3	Johnson

Rank	Candidate
1	Clinton
2	Johnson
3	Trump

To save some ink, I could also specify my preferences in abbreviated form by simply writing $C \succ T \succ J$ for the preference order on the left, and $C \succ J \succ T$ for the one on the right. Note that the $\succ$ symbol is analogous to the "greater than" sign ($>$) that we use to compare numbers; it means "is preferred to" and provides a compact way of listing preferences.

Question 2.7.*

(a) If you didn't know that Clinton was my top choice, how many possible rankings could I have had of Clinton, Trump, and Johnson?

(b) Write down each of the possible rankings from part (a). For each one, provide both a table and a list using $\succ$ notation.

(c) If I had also included Green Party candidate Jill Stein in my rankings, how many possible rankings could I have had?

(d) Suppose again that Clinton was my top choice. In how many different ways could I have ranked Clinton, Trump, Johnson, and Stein?

It's important to note that in many voting systems, only the candidate at the top of each voter's preference order matters. For instance, with plurality or majority rule, the ballot I submit only allows me to vote for my first choice. In my mind, I may not have even formulated my preferences among the other candidates. Perhaps I simply thought to myself, "I want

Clinton to win, and so I don't care about any of the other candidates." Or perhaps I was able to form a ranking of Clinton and Trump, but didn't know enough about any of the other candidates to have any real preference between them. In either case, the voting system that was used to make the decision didn't ask me for any of this additional information. It simply asked me to provide my first choice.

That said, even voting systems that use only the voters' top-ranked choices often yield a natural overall ranking of the candidates. For instance, returning to the 2016 U.S. presidential election, plurality in the state of New Hampshire produced the natural ranking $C \succ T \succ J \succ \cdots$, since Clinton received more votes than Trump, who received more votes than Johnson, who received more votes than any of the other candidates.

The ranking of the candidates produced by a voting system is called a **societal preference order**; it can be thought of as the ranking of the candidates that, according to the voting system being used, best represents the will of the voters. Of course, the winner of the election is just the first candidate listed in the societal preference order.

Question 2.8.* Suppose Filiz, Gerald, Helen, and Ivan are all running for the coveted office of President of the Cartoon Voice Actors' Association of Bulgaria (CVAAB). The preference orders of each of the 27 members of the association are given in Table 2.4. Such a table is called a **preference schedule**. The column headings indicate the number of voters with each preference order. For instance, the first column indicates that 12 members of the association have the preference order $F \succ G \succ H \succ I$. Note that only four of the many preference orders possible for this election are represented in the table.

	Number of Voters			
Rank	12	7	5	3
1	F	G	H	I
2	G	H	I	H
3	H	I	F	G
4	I	F	G	F

TABLE 2.4. Preference schedule for the CVAAB election

(a) Under majority rule, what would the outcome of the election be?

(b) Under plurality, what would the outcome of the election be, and what societal preference order would be produced?

Question 2.9. Do you think the plurality winner of the CVAAB presidential election in Question 2.8 best represents the will of the voters? If so, explain why. Otherwise, give a convincing argument for why you think some other candidate would be better.

Question 2.10. A critic of the plurality method writes the following:

> *Under plurality, it is possible for the winner of an election to be ranked first by an arbitrarily small percentage of the electorate and last by an arbitrarily large percentage of the electorate.*

Write a statement either in support of or in opposition to this view. Use preference schedules to strengthen your argument. (Hint: You may want to look back at your answer to Question 2.5.)

Back to Borda

Recall that we left the Borda count knowing only that it had the ugly ability to violate the seemingly natural majority criterion. Using the language of the previous section, the majority criterion can be defined as follows:

Definition 2.11. A voting system satisfies the **majority criterion** if whenever a candidate is ranked first by a majority of the voters, that candidate will be ranked first in the resulting societal preference order.

Question 2.12.* Do the 2016 U.S. presidential election results in the state of New Hampshire prove that the plurality method violates (i.e., does not satisfy) the majority criterion? Why or why not?

Question 2.13. Do you think that the majority criterion is reasonable? Should the Borda count be discarded because it does not satisfy the majority criterion?

So in what bizarre way does the Borda count work to make it capable of violating the majority criterion? The next definition provides the details.

Definition 2.14. Consider an election with n candidates. The **Borda count** works as follows:

- Each voter submits a ballot that contains their entire preference order for all the candidates in the election.
- For each ballot cast, points are awarded to each candidate according to the following rules:
  - A first-place ranking is worth $n - 1$ points.
  - A second-place ranking is worth $n - 2$ points.
  - A third-place ranking is worth $n - 3$ points.

    $\vdots$
  - A last-place (nth-place) ranking is worth $n - n = 0$ points.

- The candidate who accumulates the largest numbers of points from all of the ballots is declared the winner, and the societal preference order is determined by listing the candidates according to the numbers of points they receive, from largest to smallest. The Borda count produces a tie when two or more candidates receive exactly the same number of points. Candidates who receive identical point totals occupy consecutive indistinguishable positions in the societal preference order. In the event that two or more candidates are tied with the largest number of points, all such candidates are declared winners.

Question 2.15.* Under the Borda count, what would the outcome of the CVAAB presidential election from Question 2.8 be? What societal preference order would be produced?

Question 2.16. Does the definition of the Borda count seem strange or unreasonable to you? If so, explain how. Otherwise, discuss the apparent contradiction in the fact that a reasonably defined system like the Borda count can violate the majority criterion.

Question 2.17. In light of your answers to Questions 2.8, 2.9, and 2.15, who do you think should win the CVAAB presidential election? Give a convincing argument to justify your answer.

May's Theorem Revisited

In this chapter, we looked at two voting systems: plurality and the Borda count. We evaluated the fairness of these systems primarily by means of the majority criterion; we saw that plurality does satisfy the majority criterion, whereas the Borda count does not.

Let's not forget, however, that we also have three criteria from Chapter 1 that we could use to evaluate these systems: anonymity, neutrality, and monotonicity. The definition of anonymity translates naturally to situations involving more than two candidates, while neutrality and monotonicity require only slight modifications.

Definition 2.18.

- A voting system is **anonymous** if it treats all of the voters equally, meaning that if any two voters traded preference orders, the outcome of the election (and the resulting societal preference order) would remain the same.
- A voting system is **neutral** if it treats all of the candidates equally, meaning that if *every* voter switched the positions of two particular candidates in their individual preference orders, the positions of these two candidates would switch in the resulting societal preference order as well.

- A voting system is **monotone** if changes favorable only to a particular candidate in individual preference orders cannot cause that candidate to be ranked lower in the resulting societal preference order.

It's important to note that, in Definition 2.18, the phrase *changes favorable only to a particular candidate* means changes that involve nothing more than increasing that candidate's rank on one or more of the individual voters' preference ballots.

Question 2.19. Clearly explain why, in a monotone voting system, changes *unfavorable* only to a particular candidate in any number of individual preference orders cannot cause that candidate to finish *higher* in the resulting societal preference order.

Question 2.20.* Suppose the CVAAB presidential election from Question 2.8 is decided by a voting system that, given the 27 preference orders shown in Table 2.4, would produce the societal preference order $H \succ F \succ G \succ I$.

(a) If the voting system were neutral, what societal preference order would be produced if the members of the CVAAB changed their preference ballots to those shown in Table 2.5?

	Number of Voters			
Rank	12	7	5	3
1	F	G	I	H
2	G	I	H	I
3	I	H	F	G
4	H	F	G	F

TABLE 2.5. Revised CVAAB preference schedule 1

(b) Again assuming only neutrality, what can you say about the societal preference order that would be produced if the CVAAB members changed their preference ballots to those shown in Table 2.6?

	Number of Voters			
Rank	12	7	5	3
1	F	G	I	H
2	G	I	H	I
3	I	H	F	F
4	H	F	G	G

TABLE 2.6. Revised CVAAB preference schedule 2

(c) Assuming the voting system is both neutral *and* monotone, what can you say about the societal preference order that would be produced if the CVAAB members changed their preference orders to those shown in Table 2.6?

Question 2.21.

(a) Which of the properties of anonymity, neutrality, and monotonicity are satisfied by plurality? Which of these three properties are not satisfied? Give a convincing argument to justify each of your answers.

(b) Which of the properties of anonymity, neutrality, and monotonicity are satisfied by the Borda count? Which of these three properties are not satisfied? Give a convincing argument to justify each of your answers.

(c) Do either of your answers to parts (a) and (b) contradict May's Theorem? Explain.

Questions for Further Study

Question 2.22.

(a) Is plurality a quota system? Give a convincing argument to justify your answer.

(b) Does your answer to part (a) contradict Theorem 1.22? Explain. (Hint: Consider your answer to Question 2.21.)

Question 2.23. How many first-place votes must a candidate receive in order to win an election if the Borda count is used to determine the winner? Give an example using a preference schedule to justify your answer.

Question 2.24. Write a short biography of Jean-Charles de Borda, including his most important contributions both inside and outside of voting theory, and some information about his military career.

Question 2.25. Suppose that the Stickeyville Board of Commissioners is proposing that their next mayoral election be decided by the Borda count instead of plurality. Based on what you've learned in this chapter, write a formal letter to the editor of the Stickeyville Daily Review either in support of or in opposition to this proposal. Regardless of your personal feelings on the issue, your letter should discuss the pros and cons of each system and specifically address the fact that the Borda count does not satisfy the majority criterion.

Question 2.26. In the CVAAB presidential election from Question 2.8, suppose that the three voters represented in the rightmost column of Table 2.4 changed their preference orders from $I \succ H \succ G \succ F$ to $I \succ H \succ F \succ G$.

(a) Did any of the voters change their preference between just Gerald and Helen?

(b) Under the Borda count, how would the societal preference order change as a result of these individual changes?

(c) In light of your answer to Question 2.15, does anything seem strange about your answers to parts (a) and (b) above? Explain.

(d) Without moving Filiz above Gerald or Helen in any individual preference order, find a way to modify the preference schedule in Table 2.4 so that Filiz wins the election under the Borda count.

Question 2.27. Suppose a small group of voters in the CVAAB presidential election from Question 2.8 are strongly opposed to Filiz being elected as their new president. Explain how these voters could manipulate the plurality method to reduce Filiz's chances of winning. (Hint: Suppose the voters introduced and supported another candidate. Whose views should this new candidate emulate in order to hurt Filiz's candidacy the most?)

Question 2.28.

(a) Find a U.S. presidential election in which the winning candidate received a plurality but not a majority of the nationwide popular vote.

(b) Find a U.S. presidential election, other than the 2016 and 2000 elections, in which the winning candidate did not receive a plurality of the nationwide popular vote.

Question 2.29. If the Borda count had been used instead of plurality to determine the winner, who do you think would have won the 2016 U.S. presidential election in New Hampshire? Give a convincing argument to justify your answer. (Hint: You will have to make some conjectures about the preferences of those who voted for Johnson. You may need to do some research to see which of the other major candidates' views most closely resembled Johnson's.)

Question 2.30.

(a) If the Borda count had been used instead of plurality to determine the winner in Florida, who do you think would have won the 2000 U.S. presidential election? Give a convincing argument to justify your answer. (Hint: You will have to make some conjectures about the preferences of those who voted for Nader. You may need to do some research to see which of the other major candidates' views most closely resembled Nader's.)

(b) On May 19, 2004, roughly six months before the 2004 U.S. presidential election, Democratic presidential candidate John Kerry met privately with Ralph Nader, who, months earlier, had announced his candidacy in the election. Why do you think Kerry arranged this meeting? What do you think the two candidates discussed?

(c) Why do you think the Republican Leadership Council ran pro-Nader ads in the weeks prior to the 2000 U.S. presidential election?

Question 2.31. Some political scientists have speculated that if Vermont senator Bernie Sanders (an independent who caucuses with the Democrats) had been a candidate in the 2016 U.S. presidential election as an independent, he would have won a head-to-head contest against either Hillary Clinton or Donald Trump.

(a) Had Sanders been a candidate, who do you think would have won the election? Do you think the race would have been close or not? Explain.

(b) Would your answer to part (a) be different if the Borda count were used instead of plurality to determine the winner of the election in each state? Explain.

Question 2.32. On October 7, 2003, residents of the state of California voted to recall then-governor Gray Davis and replace him with Hollywood actor and former Mr. Olympia Arnold Schwarzenegger. Investigate this election, and write a detailed summary of your findings. Include in your summary answers to at least the following questions:

- How many candidates were listed on the ballot as potential replacements for Davis if he were to be recalled?
- How many votes were cast statewide? Did Schwarzenegger receive a majority of them?
- Given the number of candidates and number of votes cast statewide, and the fact that the winner was chosen using plurality, what is the smallest number of votes Schwarzenegger could have received and still have had a chance of winning the election? What is the maximum number of voters who could have preferred Schwarzenegger the *least* among the potential replacement candidates in order for him to still have had a chance of winning the election?
- What legal requirement was met by the citizens of California in order for the recall election to take place?
- What were the main arguments in favor of Davis being recalled?
- Who were some of the more famous candidates to replace Davis? What were their general views?
- How did the actual voting take place?
- What questions were on the ballot?
- What were the results for each question on the ballot?

Question 2.33. The top three teams from the Associated Press college football poll that was released on December 5, 2004 are shown in Table 2.7. The poll was formed using a version of the Borda count and was based on the votes of 65 voters, each of whom ranked 25 teams. The only difference between the version of the Borda count used to form this poll and the Borda

count as it was presented in this chapter is that, instead of awarding points for each ballot from 24 for a first-place ranking down to 0 for a last-place ranking (as described in Definition 2.14), the poll awards points for each ballot from 25 for a first-place ranking down to 1 for a last-place ranking. (This is to account for the fact that there are more than 25 teams that could be ranked, and the team with the last-place ranking on each ballot needs to be distinguished from the teams left off the ballot.) Based on this information and the information given in the Table 2.7, is it possible to conclude that all 65 voters had Southern California, Oklahoma, and Auburn ranked in some order as their top three choices? Why or why not?

Rank	Team	Points	First-Place Votes
1	Southern California	1599	44
2	Oklahoma	1556	14
3	Auburn	1525	7
⋮	⋮	⋮	(all 0)

TABLE 2.7. AP college football poll, December 5, 2004

Question 2.34. The top three teams from the Coaches' college football poll that was released on October 12, 2014 are shown in Table 2.8. This poll was formed using the same version of the Borda count as the poll described in Question 2.33, and was based on the votes of 62 voters. Does this poll illustrate a violation of the majority criterion? If so, explain why. Otherwise, state the number of additional first-place votes Florida State would have needed to receive for this ranking to illustrate such a violation.

Rank	Team	Points	First-Place Votes
1	Mississippi State	1490	26
2	Florida State	1489	31
3	Mississippi	1436	5
⋮	⋮	⋮	(all 0)

TABLE 2.8. Coaches' college football poll, October 12, 2014

Question 2.35. Consider again the Coaches' college football poll that was released on October 12, 2014, for which the top three teams are shown in Table 2.8. The reason this was called the *Coaches'* poll is because the voters in the poll were a collection of actual college football coaches, including possibly the coaches at Mississippi State and Florida State. Explain how the coaches at Mississippi State and Florida State, were they in fact voters, could have easily manipulated the results of the poll.

Question 2.36. Each year, the Most Valuable Player in major league baseball's National League is chosen by a group of 30 sportswriters using a variation of the Borda count, which differs from the Borda count as it was presented in this chapter only in the number of points assigned to the players on each ballot. The winners of the 2015 and 2016 National League MVP awards and the numbers of points/votes they received are shown in Table 2.9. From the information in the table, determine the numbers of points awarded for first and second-place votes on National League MVP ballots.

Year	Winner	Points	1st-Place Votes	2nd-Place Votes
2015	Bryce Harper	420	30	0
2016	Kris Bryant	415	29	1

TABLE 2.9. National League MVP winners, 2015 and 2016

Question 2.37. As noted in Question 2.36, major league baseball's Most Valuable Players are chosen at the end of each season by a group of sportswriters using a variation of the Borda count. Investigate the result of the American League MVP voting after the 2001 baseball season, and determine whether the result would have been the same if the Borda count had been used exactly as it was presented in this chapter.

Question 2.38. Find out how voting is conducted for the Heisman Memorial Trophy, and write a detailed summary of your findings. Include in your summary at least a description of what the Heisman Trophy is given for, who votes, how the voting is conducted, and the exact voting system used to declare the winner. In addition, use the 2008 vote results to illustrate the calculations required for this voting system, and comment on anything about these results that seems surprising or unusual to you.

Question 2.39. Find a magazine, newspaper, or web site that describes an example not related to sports where the Borda count or a version of the Borda count was used to arrive at some type of decision or ranking. Write a summary of your findings, including the name of your source, the outcome of the example, and how the Borda count was used.

Question 2.40. Decide whether each of the following statements is true or false. Give a convincing argument or example to justify each of your answers.

(a) In a three-candidate election that does not result in a tie, the Borda count winner must receive at least one first-place vote.

(b) In a four-candidate election that does not result in a tie, the Borda count winner must receive at least one first-place vote.

Question 2.41. Find a copy of the article "Would the Borda Count Have Avoided the Civil War?" by Alexander Tabarrok and Lee Spector in the

Journal of Theoretical Politics. Write a summary of the article, including the authors' answer to the question posed in the article's title.

Question 2.42. Consider an election with four candidates and the preferences shown below:

	Number of Voters		
Rank	51	25	24
1	*A*	*C*	*D*
2	*B*	*B*	*B*
3	*C*	*D*	*C*
4	*D*	*A*	*A*

(a) Who would win this election under any system that satisfies the majority criterion?

(b) Who would win this election under the Borda count?

(c) Which of the outcomes from parts (a) and (b) do you think is more fair? In your opinion, which better represents the will of the voters?

(d) Do your answers to parts (a)–(c) affect your opinion of the majority criterion in any way? Explain.

Answers to Starred Questions

2.2. (a) In an election with only two candidates, one candidate cannot receive more votes than the other candidate without receiving *more than half* of the votes. Likewise, one candidate cannot receive more than half of the votes without receiving more votes than the other candidate. Otherwise, the number of votes would exceed the number of voters.

(b) In an election with more than two candidates, one candidate could receive more votes than any of the others but still not receive *more than half* of the votes cast. The 2016 U.S. presidential election election in the state of New Hampshire is an example of this.

2.4. (a) Majority rule results in a tie whenever none of the candidates in an election receive more than half of the votes cast. For plurality to result in a tie, two or more of the candidates would have to receive the *exact same* number of votes *and* more votes than all of the other candidates. Thus, majority rule is much more likely to result in a tie.

(b) For a candidate to win an election under majority rule, they must have received more than half of the votes cast. Since none of the

other candidates could have also received more than half of the votes, the candidate who did receive more than half would have necessarily received more votes than any of the other candidates and thus be guaranteed of also winning under plurality.

(c) A plurality winner of an election need not also win under majority rule. For example, although Hillary Clinton was the plurality winner of the 2016 U.S. presidential election in New Hampshire, she would not have won under majority rule since she did not receive a majority of the votes cast.

2.6. Nebraska would have been ranked first with a majority (26 out of 50) of the first-place votes.

2.7. (a) Since there are 3 choices for my first-place candidate, 2 choices for my second-place candidate (one of the two that I didn't rank first), and only 1 choice for my third-place candidate (the one that I didn't rank either first or second), it follows that there are $3 \times 2 \times 1 = 6$ different ways in which I could have ranked the candidates.

(b) The six possible rankings are the two displayed in the text along with the four shown in the following table.

Rank	Candidate	Candidate	Candidate	Candidate
1	Trump	Trump	Johnson	Johnson
2	Clinton	Johnson	Clinton	Trump
3	Johnson	Clinton	Trump	Clinton

(c) Similar reasoning to that in part (a) applies. With four candidates, there are $4 \times 3 \times 2 \times 1 = 24$ different ways in which I could have ranked the candidates.

(d) If Clinton must be ranked first, then I would need to rank the remaining three candidates. There are $3 \times 2 \times 1 = 6$ different ways to do so.

2.8. (a) Under majority rule, the election would result in a tie (with no winner elected), since no candidate would receive more than half of the first-place votes cast.

(b) Under plurality, Filiz would win, since she would receive more first-place votes (12) than any of the other three candidates. The societal preference order would be $F \succ G \succ H \succ I$.

2.12. The 2016 U.S. presidential election results in the state of New Hampshire *do not* prove that the plurality method violates the majority criterion. The majority criterion states that if a candidate in an election receives more than half of the votes cast, then that candidate should be declared the winner. It does not, however, state that if a candidate receives less than half

of the votes, then that candidate should not be declared the winner. The plurality method actually satisfies the majority criterion. As we argued in Question 2.4, a majority rule winner would always be declared a winner under plurality as well.

2.15. According to the preference schedule in Question 2.8, Filiz would receive 12 first-place votes, 0 second-place votes, 5 third-place votes, and 10 last-place votes. Since there are 4 candidates in the election, first-place votes are worth 3 points each, second-place votes are worth 2 points, third-place votes 1 point, and last-place votes 0 points. Thus, Filiz would be awarded $(12 \times 3) + (0 \times 2) + (5 \times 1) + (10 \times 0) = 41$ points. Similar calculations show that Gerald would be awarded 48 points, Helen 47 points, and Ivan 26 points. Thus, the winner under the Borda count would be Gerald, and the resulting societal preference order would be $G \succ H \succ F \succ I$.

2.20. (a) Since all of the voters swapped the positions of H and I in their individual preference orders, neutrality dictates that H and I would have to be swapped in the resulting societal preference order, yielding $I \succ F \succ G \succ H$.

(b) Although all of the voters swapped the positions of H and I in their preference orders, the three voters represented in the rightmost column of the table also swapped F and G. Since neutrality only dictates changes to the societal preference order when each voter swaps exactly the same candidates, we have no way of knowing what the new societal preference order would be in this case.

(c) Notice first that the preference schedule in Table 2.6 differs from the one in Table 2.5 only by a swap of F and G in the three voters' preference orders represented in the rightmost column of each table, a change that is favorable only to F. By part (a), we know that the preference schedule in Table 2.5 produces the societal preference order $I \succ F \succ G \succ H$. Since the preference schedule in Table 2.6 can be obtained from the one in Table 2.5 by changes favorable only to F (and unfavorable to G), monotonicity requires that F remain preferred to G and H in the resulting societal preference order. However, there are many societal preference orders that would be consistent with these requirements. To list a few:

$$I \succ F \succ G \succ H,$$
$$F \succ I \succ G \succ H,$$

and even the somewhat strange

$$F \succ I \succ H \succ G.$$

(Note the reversal of the ranking of G and H in this last order.)

Chapter 3

Back into the Ring

Focus Questions

In this chapter, we'll explore the following questions:

- What is the Condorcet winner criterion? Which voting systems satisfy the Condorcet winner criterion, and which do not?
- What is sequential pairwise voting, and how does it work?
- What is instant runoff voting? How does it work, and where is it commonly used?
- What are some of the advantages and disadvantages of sequential pairwise voting and instant runoff?

Warmup 3.1. Suppose Skip, Norm, and Jesse are all running for President of the 10,000 Lakes Club, with the preferences of the 100 members of the club as shown in Table 3.1.

	Number of Voters			
Rank	35	28	20	17
1	N	S	J	J
2	S	N	N	S
3	J	J	S	N

TABLE 3.1. Preference schedule for the 10,000 Lakes Club

(a) What would be the outcome of the election under majority rule?

(b) What would be the outcome of the election under plurality?

(c) What would be the outcome of the election under the Borda count?

(d) Which candidate is ranked first by the largest number of voters?

(e) Which candidate is ranked last by the largest number of voters?

(f) In a head-to-head contest[1] between just Skip and Norm, who would win?

(g) In a head-to-head contest between just Skip and Jesse, who would win?

(h) In a head-to-head contest between just Norm and Jesse, who would win?

(i) Does anything about your answers to parts (a)–(h) above strike you as being strange or unusual? Explain.

As you probably noticed, the election from Warmup 3.1 exhibits a number of peculiarities. For one thing, the plurality winner, Jesse, is ranked first by only 37% of the voters. The other 63% rank him last; they would all prefer either Skip or Norm. Furthermore, Jesse would lose to either of the other two candidates if he went up against them head-to-head.

It may be tempting to dismiss these observations as features of a contrived example that would never occur in real life... unless you're from Minnesota or are familiar with the state's 38th governor, former professional wrestler and radio shock-jock Jesse "The Body" Ventura. In 1998, Ventura, running as a Reform Party candidate, claimed a stunning victory over Minnesota Attorney General Skip Humphrey (a Democrat) and St. Paul Mayor Norm Coleman (a Republican) in the state's gubernatorial race. Ventura won under plurality by receiving 37% of the popular votes, higher than the percentages received by either Humphrey (28%) or Coleman (35%). Although it is impossible to know exactly how Minnesota's voters ranked Ventura in comparison to the other candidates, many have speculated that the voters' preferences looked a lot like those in Table 3.1, with a large number of voters ranking Ventura in last place.[2]

For a more recent example, consider the 2016 Republican primaries in the U.S. presidential election. One public opinion poll, conducted in March of 2016, suggests that Donald Trump would have won a plurality election against Ted Cruz, John Kasich, and Marco Rubio, but would have lost to each of these candidates—by double-digit margins—if he went up against them head-to-head [**31**].

Outside the U.S., the plurality winner in the initial round of voting in the 2017 French presidential election received less than 25% of the votes cast, with the second-, third-, and fourth-place candidates all trailing by less than 5%. However, in spite of the close outcome in this initial round, the winner, Emmanuel Macron, handily defeated his opponent, Marine Le Pen, in the second round, winning by a 2-to-1 (66% to 34%) margin.

[1] By *head-to-head contest* we mean a two-candidate election with the winner decided by majority rule. Thus, when we say that A would defeat B in a head-to-head contest, we mean that if the voters were forced to choose between only these two candidates, then A would receive more votes than B.

[2] For a well-researched opposing view, see [**32**].

Now back to Minnesota: In the weeks and months following the 1998 gubernatorial election, many political commentators tried to explain how Ventura, whose only prior political experience had been a four-year stint as mayor of Brooklyn Park, Minnesota, was able to defeat two well-known opponents, each having significantly more experience in the political arena. Many suggested (correctly) that Ventura had mobilized more young voters than either of the other two candidates. Others speculated that Ventura's celebrity and larger-than-life personality had earned him the votes of those who were not familiar with the political views of any of the three candidates. Only a handful, however, suggested that Ventura's victory might have been merely a consequence of the voting system that was used: plurality.

This, of course, is what we are most interested in. In Warmup 3.1, we saw how an alternative to plurality—the Borda count—might have produced an outcome that represented the will of Minnesota's voters better than the plurality outcome. But, as we saw in the last chapter, the Borda count is not without flaws; for example, it is capable of violating the majority criterion, a desirable property that even plurality satisfies.

As we're beginning to see, deciding elections with more than two candidates can be tricky. There are a lot of thorny issues to deal with, and we'll have to wrestle with these issues for a while (pun definitely intended) if we are to have any hope of resolving them. That's exactly what we'll do in this chapter.

Condorcet Winners and Losers

In Warmup 3.1, we made a couple of important but troublesome observations. First, we saw that plurality can fail to elect a candidate who would win a head-to-head contest against each of the other candidates. Second, and even more disturbing, we saw that plurality can elect a candidate who would *lose* a head-to-head contest against each of the other candidates.

These kinds of phenomena are often attributed to the well-named Marie Jean Antoine Nicolas de Caritat, the Marquis de Condorcet, a French mathematician and contemporary of Borda who is usually referred to simply by the name *Condorcet*. The following terms, which correspond to the ideas described above, were named in honor of Condorcet.

Definition 3.2.

- A **Condorcet winner** is a candidate in an election who would win a head-to-head contest (with the winner decided by majority rule) against each of the other candidates.
- A **Condorcet loser** is a candidate in an election who would lose a head-to-head contest (with the winner decided by majority rule) against each of the other candidates.

- A voting system that will always elect a Condorcet winner, whenever one exists, is said to satisfy the **Condorcet winner criterion** (**CWC** for short).
- A voting system that will never elect a Condorcet loser is said to satisfy the **Condorcet loser criterion** (**CLC** for short).

Question 3.3.* Assuming the data in Table 3.1 accurately reflects the preferences of the voters in the 1998 Minnesota gubernatorial election, was there a Condorcet winner and/or loser in the election? If so, who?

Knowing that a voting system satisfies the Condorcet loser criterion is useful; after all, such a system would avoid the possibility of electing a candidate who would consistently lose in head-to-head contests. But why should we stop there? After all, the Condorcet winner criterion seems entirely reasonable and is perhaps even more important because it declares a condition that has the potential to yield a winner, whereas the Condorcet loser criterion simply rules out certain candidates. And if there is a candidate in an election who would win a head-to-head contest against any of their opponents, why shouldn't that candidate be elected? With this in mind, our goal for the next few pages will be to find a voting system that satisfies the CWC—and hopefully the other desirable properties we've considered as well. Before we do so, however, let's pause to consider a few other important details.

Question 3.4.* Consider the preference schedule in Table 3.2.

	Number of Voters		
Rank	1	1	1
1	A	B	C
2	B	C	A
3	C	A	B

TABLE 3.2. Condorcet's preference schedule

(a) In a head-to-head contest between just candidates A and B, who would win?

(b) In a head-to-head contest between just B and C, who would win?

(c) In a head-to-head contest between just A and C, who would win?

(d) Does anything about your answers to parts (a)–(c) strike you as being strange or unusual? Explain.

(e) Is there a Condorcet winner and/or loser in this election? Explain.

Question 3.4 shows that it is possible for an election to have neither a Condorcet winner nor a Condorcet loser. This explains why the phrase

"when one exists" is a necessary part of the definition of the Condorcet winner criterion. But let's suppose for the moment that we are considering an election in which there is a Condorcet winner. Can there be more than one?

Question 3.5.* Assume that, in a certain election, there are two different Condorcet winners. Explain why this assumption leads to a contradiction to the definition of a Condorcet winner, and why this contradiction shows that it is impossible for an election to have more than one Condorcet winner.

Question 3.6. Use the same kind of reasoning as in Question 3.5 to explain why it is impossible for an election to have more than one Condorcet loser.

We're now ready to look for a voting system that satisfies the Condorcet winner criterion. Let's begin with the systems we investigated in Chapters 1 and 2. As we saw in Warmup 3.1, plurality is not a good option if we want to elect Condorcet winners and avoid electing Condorcet losers. But what about majority rule or the Borda count?

Question 3.7.*

(a) Explain why, whenever majority rule does not result in a tie, the majority rule winner will be a Condorcet winner.

(b) Does your answer to part (a) imply that majority rule satisfies the Condorcet winner criterion? If so, explain why. Otherwise, give an example to show that majority rule can violate the CWC.

(c) Does your answer to part (a) imply that majority rule satisfies the Condorcet loser criterion? If so, explain why. Otherwise, give an example to show that majority rule can violate the CLC.

(d) Are there any special types of elections for which majority rule does satisfy the CWC? Give a convincing argument to justify your answer.

(e) Use your answer to part (a) to explain why any voting system that violates the majority criterion must also violate the CWC.

(f) Use your answer to part (e) to explain why the Borda count violates the CWC.

So it looks as if the well has run dry. We are going to have to invent or discover some new voting system if we are to have a chance of satisfying the elusive Condorcet winner criterion. But first let's take a moment to formalize the observation we made in part (e) of Question 3.7. What we said there was that any voting system that violates the majority criterion must also violate the CWC. Likewise, we could also say that any voting system that satisfies the CWC must also satisfy the majority criterion. In other words, the Condorcet winner criterion is a stronger condition than the majority criterion. For reference, we'll state these observations in the following theorem.

Theorem 3.8.

- *If a candidate in an election receives a majority of the first-place votes, then that candidate will be a Condorcet winner.*
- *If a voting system satisfies the Condorcet winner criterion, then it will also satisfy the majority criterion.*
- *If a voting system violates the majority criterion, then it will also violate the Condorcet winner criterion.*

Now let's see if we can find a new voting system that satisfies the Condorcet winner criterion.

Sequential Pairwise Voting

Since the definition of a Condorcet winner involves head-to-head, two-candidate elections, a good bet for a system that satisfies the Condorcet winner criterion would be one that uses head-to-head contests to determine the winner. Since we are considering elections with more than two candidates, we'll obviously need to hold more than just a single two-candidate election. But perhaps if we conducted a *sequence* of two-candidate elections, we would be able to collect enough information to determine a winner. If we were lucky, this winner would be the Condorcet winner for the election, just as we wanted.

To illustrate one such method, let's return to the CVAAB presidential election from Question 2.8. Recall that the preferences of the 27 members of the CVAAB are as shown in Table 3.3.

	Number of Voters			
Rank	12	7	5	3
1	F	G	H	I
2	G	H	I	H
3	H	I	F	G
4	I	F	G	F

TABLE 3.3. Preference schedule for the CVAAB election

Remember also that, for this election, the plurality method produced the societal preference order $F \succ G \succ H \succ I$, whereas the Borda count produced $G \succ H \succ F \succ I$. Let's now see what societal preference order would be produced by a sequence of head-to-head contests. Here's how we'll run the election:

Step 1: First, we'll ask the voters to choose between just Gerald and Helen. Since this is a two-candidate election, we'll use majority rule to decide the winner.

Step 2: Next, we'll ask the voters to choose between Filiz and the winner from Step 1, again using majority rule to decide the winner.

Step 3: Finally, we'll ask the voters to choose between Ivan and the winner from Step 2. Whoever wins this third head-to-head contest will be declared the overall winner of the election.

Question 3.9.*

(a) Under the method described in Steps 1–3 above, who would win the CVAAB presidential election?

(b) Under the method described in Steps 1–3 above, what societal preference order do you think would best represent the will of the voters in the election?

(c) In light of the plurality and Borda count results for this election, does anything about your answers to parts (a) and (b) strike you as being strange or unusual? Explain.

(d) Is there a Condorcet winner and/or loser in this election? Explain.

While your answer to Question 3.9 may not shed a lot of light on who really should be elected as the next CVAAB president, it does at least illustrate a way to use head-to-head contests to determine the winner of an election with more than two candidates. This voting system is known as **sequential pairwise voting**.

Before going on, we need to stop for a moment and consider one very important difference between sequential pairwise voting and the other systems we've already considered. Notice that with plurality and the Borda count, we had no problems constructing societal preference orders based on the results of the election. Because of this, you may have assumed that you could do the same thing in part (b) of Question 3.9. In fact, you might have assembled the results of each head-to-head contest to arrive at the societal preference order $I \succ F \succ G \succ H$. This ordering seems quite natural and is consistent with the results of each head-to-head contest we conducted. But what about the pairs of candidates that didn't compete directly against each other—for example, Ivan and Gerald? It would seem natural to conclude that if Ivan beats Filiz and Filiz beats Gerald, then Ivan should certainly beat Gerald. As it turns out, however, this is not always the case; in this example, Ivan would actually *lose* to Gerald (by a whopping margin!) if the voters were asked to decide between just the two of them. Look back to Question 3.4 to see another example of this type of unexpected behavior. Something similar happened there, and you may have even pointed it out in your answer to part (d) of that question.

What you've probably noticed by now is that voting theory is full of mind-boggling anomalies, which we often call voting **paradoxes**.[3] The example in Question 3.4 is called **Condorcet's paradox** and is quite famous in the world of election perversities.

Returning to our discussion of societal preference orders, the point that must be made is that sequential pairwise voting does not always produce a well-defined societal preference order. There are several ways in which we could deal with this, but for now we'll handle it by simply defining societal preference orders under sequential pairwise voting so that the winning candidate is ranked first and all of the other candidates are tied for second (or last, depending on how you look at it). In the example from Question 3.9, we would represent this order by $I \succ F \approx G \approx H$. Notice that, in the same way we used a fancy version of the greater than symbol to indicate a preference between two candidates ($\succ$ instead of $>$), we use a fancy version of the equals symbol to indicate a tie between two candidates ($\approx$ instead of $=$). And from now on we'll use this notation in cases where plurality or the Borda count or any other voting system results in a tie between candidates in a societal preference order.

And now one last detail: Those of you who are really on your toes might be somewhat skeptical of the solution we just proposed. After all, even the preference order $I \succ F \approx G \approx H$ suggests that I would beat G in a head-to-head contest, which we know not to be the case. If you're thinking this, you're absolutely right—we haven't completely solved the problem that we set out to solve. But we have made the situation somewhat better by eliminating the potentially false comparisons between losing candidates. Also, remember that even though G would beat I in a head-to-head contest, I is the unique winner under sequential pairwise voting. In that sense, I really is preferred to G, just as Ventura was preferred to both Coleman and Humphrey in the 1998 Minnesota gubernatorial election (as decided by plurality), even though either of these opponents might have been able to defeat him in a head-to-head contest. We can see then that, as is the case with many voting systems, there is more to sequential pairwise voting than what first meets the eye. As the late radio commentator Paul Harvey used to say, in a few moments you'll know "the rest of the story." But first let's see how sequential pairwise voting stacks up with regard to Condorcet's properties.

Question 3.10.*

(a) Could a Condorcet winner ever lose a head-to-head contest with another candidate? Why or why not?

(b) What does your answer to part (a) allow you to conclude about sequential pairwise voting and the Condorcet winner criterion?

[3] Webster's Dictionary defines a paradox to be a tenet or proposition contrary to received opinion; an assertion or sentiment seemingly contradictory, or opposed to common sense; that which in appearance or terms is absurd, but yet may be true in fact.

Question 3.11. Does sequential pairwise voting satisfy the Condorcet loser criterion? If so, explain why. Otherwise, give an example of a preference schedule for which sequential pairwise voting would elect a Condorcet loser.

So, apart from the problem in defining societal preference orders, is sequential pairwise voting a good system for choosing the winner of an election with more than two candidates? In at least one regard, it looks quite promising. After all, sequential pairwise voting will *always* elect a Condorcet winner when one exists. But what happens in situations where there is no Condorcet winner, as was the case in the CVAAB presidential election?

To answer this question, first observe that in sequential pairwise voting, we must specify—before the election takes place—the order in which the candidates will compete against each other. This seemingly harmless sequence is called the **agenda** and is usually specified by simply listing the candidates in the order in which they are to be introduced into the comparisons. For example, we used the agenda G, H, F, I in the CVAAB presidential election from Question 3.9.

Question 3.12.* Who would win the CVAAB presidential election using sequential pairwise voting with the agenda F, G, H, I?

Question 3.13.

(a) Find a sequential pairwise voting agenda for which Filiz would win the CVAAB presidential election.

(b) Find a sequential pairwise voting agenda for which Gerald would win the CVAAB presidential election.

By now, you should be convinced that sequential pairwise voting, although avoiding some pitfalls, has at least one significant problem of its own. Questions 3.12 and 3.13 illustrate how, in the absence of a Condorcet winner, the agenda can play an inordinately powerful role in determining the winner of the election.[4] This property indicates that sequential pairwise voting is highly manipulable, a fact that has not gone unnoticed by politicians and other decision-makers who rely on it. Because the agenda can give an advantage to specific candidates—while putting others at a disadvantage—it is reasonable to suspect that sequential pairwise voting may also violate the fundamental property of neutrality. Unfortunately, our next question confirms this suspicion.

[44]Incidentally, there is an obvious similarity between the lose-once-and-you're-out philosophies of sequential pairwise voting and single-elimination tournaments in which a participant can be declared the winner without having to win head-to-head contests against all or even most of the other participants. The fact that sequential pairwise voting is highly dependent on the agenda chosen is illustrated by the fact that in many single-elimination tournaments, the participants expected to perform the best are often "seeded" in such a way that they would meet each other as late as possible in the tournament.

Question 3.14. Suppose that all of the voters in the CVAAB presidential election switched the positions of I and H in their preference orders, yielding the new preference schedule shown in Table 3.4.

	Number of Voters			
Rank	12	7	5	3
1	F	G	I	H
2	G	I	H	I
3	I	H	F	G
4	H	F	G	F

TABLE 3.4. Revised CVAAB preference schedule

(a) Using sequential pairwise voting with the agenda G, H, F, I, what societal preference order would result from this new preference schedule?

(b) Explain why your answers to Question 3.9 and part (a) of this question show that sequential pairwise voting is not neutral.

So what should we do? Should we give up on sequential pairwise voting, or does it perhaps have some redeeming qualities that make it worth our consideration? Since sequential pairwise voting does arise naturally in a number of important situations, one would assume that it has some desirable features that compensate for its flaws. The Condorcet winner criterion is one such feature—and in elections with a Condorcet winner, the neutrality issue is moot, since the Condorcet winner will be selected regardless of the agenda. Moreover, it turns out that sequential pairwise voting is both anonymous and monotone.

Question 3.15. Explain why sequential pairwise voting is both anonymous and monotone.

Finally, Theorem 3.8 tells us that sequential pairwise voting also satisfies the majority criterion. And since each voting system we've considered for elections with more than two candidates violates at least one of the desirable properties we've investigated, one could argue that sequential pairwise voting is just as good as any of the others. Of course, it could also be the case that we just haven't let the cat out of the bag yet. Perhaps we just need to look a little bit further to find a voting system that will put an end to this whole discussion.

Instant Runoff

In the mid-1800s, an English lawyer and political reformist named Thomas Hare proposed a voting system that involved successively eliminating candidates until only one, the winner, remained. This system, now known as **single transferable vote**, or **instant runoff**, was well-received at the time

and has grown in popularity since. Instant runoff is currently used in some important national elections worldwide—for example, to elect the presidents of India and Ireland, and members of the Australian House of Representatives. Instant runoff is also used to elect the mayors of several large cities, including London, San Francisco, and Minneapolis. According to FairVote, a nonpartisan organization that advocates for electoral reform, "literally hundreds of jurisdictions, organizations and corporations use instant runoff voting to elect leaders."

Does that sound convincing? Well, there's more. In 1860, philosopher John Stuart Mill (who we should admit was a close friend of Hare's) wrote the following in his book *Considerations on Representative Government*:

> *The more [the details of the Hare system] are studied the stronger, I venture to predict, will be the impression of the perfect feasibility of the scheme, and its transcendent advantages. Such and so numerous are these, that, in my conviction, they place Mr. Hare's plan among the very greatest improvements yet made in the theory and practice of government.*

It would be hard to find a much stronger endorsement than this, so let's investigate instant runoff and see if it really is as good as John Stuart Mill and the folks at FairVote would have us believe.

Definition 3.16. The **instant runoff** voting system works according to the following three steps:

Step 1: Each voter in the election submits their entire preference order.

Step 2: The candidate with the fewest first-place votes (or candidates in the case of a tie) is eliminated from each voter's preference order, and the remaining candidates are moved up on each preference order, yielding a new collection of preferences.

Step 3: Step 2 is repeated until only a single candidate remains. This candidate is then declared the winner of the election.

If desired, a societal preference order for the election can be formed by listing the candidates in the reverse of the order in which they were eliminated—that is, by listing the candidates starting with the winner, followed by the last candidate eliminated, and continuing down to the first candidate eliminated.

Question 3.17.* Suppose Amaya, Brandon, Carlos, and Delilah are all running for the position of Chair of the Mathematics Department at Podunk University. The preferences of the 21 members of the department are shown in Table 3.5.

(a) Which candidate would be eliminated first under the instant runoff method? Which would be eliminated second? Third?

	Number of Voters			
Rank	7	6	5	3
1	A	B	C	D
2	B	A	B	C
3	C	C	A	B
4	D	D	D	A

TABLE 3.5. Preference schedule for PU math chair election

(b) Who would win the election under the instant runoff method? What would be the resulting societal preference order?

Now let's check the properties. We'll start with anonymity and neutrality. As with some of the systems we've considered already, it might seem fairly obvious to you that instant runoff is both anonymous and neutral. After all, nothing in its definition mentions anything about individual voters or candidates. This is in contrast to sequential pairwise voting, which uses an agenda that specifically lists each candidate, and dictatorships, which explicitly designate one of the voters as being more important than all of the others. Nevertheless, we've seen our intuition fail us before, so let's refer back to the definitions of anonymity and neutrality to explain why instant runoff satisfies both of these properties.

Question 3.18. Use Definition 2.18 to write a detailed explanation of why instant runoff is both anonymous and neutral.

The definition of monotonicity is slightly more complicated, so we'll consider an example before formulating a general argument.

Question 3.19.* Consider again the election for the PU Mathematics Department Chair from Question 3.17. Suppose that, after a heated debate, the three voters from the rightmost column of Table 3.5 change their preferences to $A \succ D \succ C \succ B$. Note that this is a change that is favorable only to Amaya.

(a) With these new preferences, who would win the election under the instant runoff method?

(b) Compare your answer to part (a) of this question with your answer to part (b) of Question 3.17. What conclusions can you draw?

Hold on a second! Did you catch what just happened in Question 3.19? How is it that instant runoff, which was supposed to save the day, doesn't even satisfy monotonicity? What's going on here? We'll have to wait to answer that question until the next chapter, but first let's see if there is anything positive we can salvage from Mr. Hare's "perfectly feasible" system.

Question 3.20.

(a) Explain why if, at any stage in the process of instant runoff voting, one candidate receives a majority of the first-place votes, then that candidate can automatically be declared the winner of the election.

(b) Use your answer to part (a) to explain why instant runoff satisfies the majority criterion.

Question 3.21. Consider an election between three candidates with the preference schedule shown in Table 3.6.

	Number of Voters		
Rank	1	2	2
1	*A*	*B*	*C*
2	*B*	*A*	*A*
3	*C*	*C*	*B*

TABLE 3.6. Instant runoff and the CWC

(a) Is there a Condorcet winner for this election?

(b) Who would win the election under the instant runoff method?

(c) Does instant runoff satisfy the Condorcet winner criterion? Use your answers to parts (a) and (b) to explain how you know.

Putting It All Together

Question 3.22.* Summarize what you've learned about voting systems for more than two candidates by completing the following table. In this table, we've used the abbreviation "MC" for the majority criterion. For the entries in the table, by "Yes" we mean that the voting system labeling the row of the entry satisfies the property labeling the column, and by "No" we mean that the voting system labeling the row violates the property labeling the column.

	Anonymous	Neutral	Monotone	MC	CWC
Plurality	Yes	Yes	Yes	Yes	No
Borda Count					
Sequential Pairwise					
Instant Runoff					

Question 3.23.* Consider again the CVAAB presidential election from Question 2.8. Who would win the election under the instant runoff method?

Question 3.24. Write a professional letter to the editor of *Squeaks and Squawks* (the official newsletter of the CVAAB) expressing your opinion about which voting system you think should be used to elect the next president of the CVAAB. Use the results of your investigations in this and the previous chapter to give a convincing argument as to why you think the system you've chosen is the most reasonable and would best represent the will of the voters. Be sure to discuss the pros and cons of your proposed system, carefully comparing and contrasting it with the other potential options.

Questions for Further Study

Question 3.25. Answer *true* or *false* for each of the following statements, and give a convincing argument to justify each of your answers.

(a) If a voting system satisfies the Condorcet winner criterion, then it must also satisfy the Condorcet loser criterion.

(b) If a voting system satisfies the Condorcet loser criterion, then it must also satisfy the Condorcet winner criterion.

Question 3.26. Consider again the election for the PU Mathematics Department Chair from Question 3.17.

(a) Who would win the election under the Borda count?

(b) In light of your answers to Question 3.17 and part (a) of this question, who do you think should be declared the winner of the election? Give a convincing argument to justify your answer.

(c) Is there an agenda under which Amaya would win using sequential pairwise voting? What about Carlos? Delilah? Explain your answers.

Question 3.27. In part (f) of Question 3.7, you explained why the Borda count violates the Condorcet winner criterion without providing an actual example to illustrate this fact. Construct an actual example (one involving a preference schedule) to show that the Borda count can violate the CWC.

Question 3.28. Does the Borda count satisfy the Condorcet loser criterion? If so, explain why. Otherwise, construct an example (one involving a preference schedule) to show that the Borda count can violate the CLC.

Question 3.29. Does instant runoff satisfy the Condorcet loser criterion? If so, explain why. Otherwise, construct an example (one involving a preference schedule) to show that instant runoff can violate the CLC.

Question 3.30. Write a short biography of the Marquis de Condorcet, including his most important contributions both inside and outside of voting theory, some information about his political importance in the French Revolution, and some information about his death.

Question 3.31. Write a short biography of Thomas Hare, including his most important contributions both inside and outside of voting theory.

Question 3.32. Investigate the arguments presented on the web site of FairVote (`http://www.fairvote.org`) in favor of the instant runoff voting system. Write a summary of your findings, including a comparison of the arguments presented there with our investigations in this chapter.

Question 3.33. Investigate the results of the 1991 Louisiana gubernatorial election, and write a summary of your findings. Include in your summary a brief description of the three most prominent candidates, their platforms and personal situations, and the eventual winner of the election. Also describe whether you think there could have been a Condorcet winner and/or loser in the election, and which candidate you think best represented the will of the voters. Clearly explain your reasoning, and use actual data from the election to help justify your claims.

Question 3.34.

(a) Find out how voting is conducted to determine the president of France, and write a summary of your findings. Include in your summary a detailed description of the voting system used to declare the winner, and how this system differs slightly from one of the systems we studied in this chapter.

(b) Investigate the results of the 2017 French presidential election, and write a summary of your findings. Include in your summary a brief description of the four candidates who received the largest numbers of votes, the platforms and personal situations of these candidates, and the eventual winner of the election.

(c) Investigate the results of the 2002 French presidential election, and write a summary of your findings. Include in your summary a brief description of the three candidates who received the largest numbers of votes, the platforms and personal situations of these candidates, and the eventual winner of the election.

Question 3.35.

(a) Find out how voting is conducted to determine the host city for the Olympic Games, and write a summary of your findings. Include in your summary a description of how the nominees are selected, who votes, how the voting is conducted, the voting system that is used to declare the winner, and how this system differs slightly from one of the systems we studied in this chapter.

(b) Investigate the results of the voting that was held by the International Olympic Committee (IOC) to determine the host city for the 2012 Summer Olympic Games from the five finalists. Write a summary of your findings, including the finalists, the winner, and the results from each round of voting.

(c) Investigate the results of the voting that was held by the IOC to determine the host city for the 2000 Summer Olympic Games from the five finalists. Write a summary of your findings, including the finalists and winner, the results from each round of voting, and the reason why one voter chose to abstain from voting in the last two rounds.

Question 3.36. Investigate the process through which candidates are nominated to receive one of the major Academy Awards (the "Oscars"), and write a detailed summary of your findings. (Note: There are two stages in the process for being selected to receive an Academy Award—the nomination stage and the final balloting to determine the winner. We are not asking for a description of the final balloting; this is done by plurality. We are asking for a description of the *nomination* stage.)

Question 3.37. Investigate Coombs' voting system, and write a summary of your findings. Include in your summary a description of how the system works, the similarities and differences between Coombs' system and the other voting systems we have studied, which of the criteria we have discussed for evaluating voting systems (anonymity, neutrality, monotonicity, the majority criterion, the CWC, and the CLC) Coombs' system satisfies, and which it violates.

Question 3.38. Research the voting system used by the reality TV show *Survivor*, and write a detailed summary of your findings. Include in your summary a comparison of this voting system with the other systems we have investigated (including Coombs' system from Question 3.37) and an analysis of the voting system according to the fairness criteria we have developed.

Question 3.39. Come up with a preference schedule for an election with four candidates in which plurality, the Borda count, sequential pairwise voting (with some agenda you specify), and instant runoff would all yield different outcomes, *and* for which the outcome of one of these four methods would demonstrate a violation of the Condorcet loser criterion.

Question 3.40. For each of the following statements, give a brief argument either for or against the statement. Support your argument with a specific example, such as a preference schedule that illustrates or refutes the claim given in the statement.

(a) The Borda count is easier to manipulate than instant runoff voting.

(b) The Borda count is less likely than other methods to elect a *consensus* or *compromise* candidate—that is, a candidate who is acceptable to a large portion of the electorate (even if relatively few voters rank them first).

(c) In certain situations, it is possible that no matter what voting method is used, and no matter which candidate is elected, some

other candidate will be preferred by a substantial majority of the voters.

(d) There are situations in which the Condorcet winner may not be the most socially desirable outcome.

(e) Plurality is more likely than other voting systems to elect a Condorcet loser.

Question 3.41. Go to FairVote's web site (`http://www.fairvote.org`) and search for "instant runoff and monotonicity." You should find a page that argues that the potential for instant runoff to violate monotonicity is not a fatal flaw. Summarize and critique this argument.

Question 3.42. In 2016, voters in the state of Maine passed a referendum that would have made Maine the first state in the U.S. to adopt instant runoff for gubernatorial elections and for congressional elections at both the federal and the state levels. Research some of the legal drama that affected the implementation of the law, and write a detailed summary of your findings. Has Maine actually implemented instant runoff voting yet?

Answers to Starred Questions

3.3. Assuming the data in Table 3.1 is correct, Norm Coleman was a Condorcet winner and Jesse Ventura was a Condorcet loser.

3.4. In head-to-head contests, A would beat B, B would beat C, and C would beat A, each by a vote of 2 to 1. This is indeed strange, since we would normally assume that if A beats B and B beats C, then A should beat C. There is neither a Condorcet winner nor a Condorcet loser.

3.5. If an election had two different Condorcet winners, say candidates A and B, then in a head-to-head contest between A and B, A would have to beat B *and* B would have to beat A. Since this obviously could never occur, it is impossible for an election to have more than one Condorcet winner.

3.7. (a) If majority rule does not result in a tie, then there must be one candidate in the election who is ranked first by more than half of the voters. Even without the votes of any of the other voters in the election, this candidate would win a head-to-head contest against any of the other candidates.

(b) Part (a) does *not* imply that majority rule satisfies the Condorcet winner criterion. To construct an example showing that majority rule can violate the CWC, consider an election with three candidates in which there is a Condorcet winner but no candidate receives a majority of the first-place votes.

(c) Part (a) does imply that majority rule satisfies the Condorcet loser criterion, since a Condorcet loser can never receive a majority of the first-place votes. (Can you explain why this is true?)

(d) Think back to the type of elections we looked at in Chapter 1.

(e) Part (a) establishes that if a candidate in an election receives a majority of the first-place votes, then that candidate will be a Condorcet winner. Thus, if a voting system ever fails to elect a majority winner, it will have also failed to elect a Condorcet winner.

(f) Since the Borda count violates the majority criterion, it must, by part (e), violate the CWC.

3.9. (a) In Step 1, Gerald would win. He would then lose to Filiz in Step 2, who would then lose to Ivan in Step 3. Thus, Ivan would win the election.

(b) Based on your answer to part (a), you might be tempted to say that the best societal preference order would be $I \succ F \succ G \succ H$. However, you may want to consider whether Ivan would beat Gerald in a head-to-head contest.

(c) Ivan being elected does seem strange, since he was ranked last in the societal preference orders produced by both plurality and the Borda count.

(d) There is neither a Condorcet winner nor a Condorcet loser.

3.10. A Condorcet winner, by definition, could never lose a head-to-head contest with another candidate. Thus, a Condorcet winner, if one exists, will always advance to the next round in sequential pairwise voting and ultimately be declared the winner. From this we can conclude that sequential pairwise voting satisfies the Condorcet winner criterion.

3.12. Helen would win.

3.17. Delilah would be eliminated first, followed by Brandon and then Carlos. Thus, Amaya would win the election, and the resulting societal preference order would be $A \succ C \succ B \succ D$.

3.19. With these new preferences, Brandon would win the election. This implies that instant runoff is not monotone(!), since a change favorable only to Amaya on individual preference ballots would cause her to go from winning the election to losing.

3.22. The table can be completed as follows:

	Anonymous	Neutral	Monotone	MC	CWC
Plurality	Yes	Yes	Yes	Yes	No
Borda Count	Yes	Yes	Yes	No	No
Sequential Pairwise	Yes	No	Yes	Yes	Yes
Instant Runoff	Yes	Yes	No	Yes	No

3.23. Helen would win.

Chapter 4

Trouble in Democracy

Focus Questions

In this chapter, we'll explore the following questions:

- What is the independence of irrelevant alternatives (IIA) criterion? Which voting systems satisfy this criterion, and which do not?
- What five conditions did Kenneth Arrow think every reasonable voting system should satisfy? What does Arrow's Theorem say about voting systems that satisfy all five of these conditions?
- What are some of the implications of Arrow's Theorem? How is Arrow's Theorem related to the search for a perfect voting system?
- What is Pareto's unanimity condition? How is unanimity related to Arrow's Theorem?

Warmup 4.1. In 1958, Duncan Black, an economist, proposed the following system for deciding the winner of an election with more than two candidates:

- Each voter submits their entire preference order, including all of the candidates in the election.
- If, based on these preference orders, a Condorcet winner exists, then this Condorcet winner is declared the overall winner of the election.
- If no Condorcet winner exists, then the Borda count is used to determine the overall winner of the election.

We have discussed a number of criteria for evaluating voting systems, including anonymity, neutrality, monotonicity, the majority criterion, and the CWC. Which of these criteria does Black's system satisfy, and which does it violate? Explain your answers in detail, and give convincing arguments (including examples where appropriate) to justify your claims.

Independence of Irrelevant Alternatives

So what do you think of Black's system? Before you answer, you may want to consider the following question.

Question 4.2.* Suppose Dale, Paul, and Wayne are the three finalists in the "World's Sexiest Man" contest held aboard the luxury cruise ship *Floater of the Seas*. Suppose also that the fifteen judges for the contest rank the finalists as shown in the preference schedule in Table 4.1.

	Number of Voters		
Rank	7	6	2
1	*P*	*D*	*W*
2	*D*	*W*	*P*
3	*W*	*P*	*D*

TABLE 4.1. World's Sexiest Man rankings

(a) Under Black's system, who would win the title of World's Sexiest Man? What ranking (societal preference order) of Dale, Paul, and Wayne would be produced by Black's system?

(b) Suppose that after the votes are cast, but before the winner is announced, Wayne is kicked off the ship for disorderly conduct, thus rendering him ineligible as a contestant. Given your answer to part (a), should Wayne's exclusion from the contest change its outcome?

(c) Suppose Wayne's name is removed from each of the fifteen ballots shown in Table 4.1, and the remaining contestants are moved up whenever necessary so that each ballot contains only a first- and a second-place candidate. What outcome would be produced by Black's system with this new collection of two-candidate ballots?

(d) Does anything about your answer to part (c) strike you as being strange or unusual? Explain.

Question 4.2 shows us that for all of the strengths of Black's system, it still has one major weakness: The removal of a candidate (Wayne) who stands little or no chance of winning the contest nevertheless has the potential to change its outcome. Because this is true, in this case we would call Wayne a *spoiler candidate*.[1] If we were particularly cynical, we might have even thought that Wayne's very participation in the contest was a sham, especially if we found out that he was a close friend of Dale's. Of course,

[11]We need only look back to the 2000 U.S. presidential election (described briefly in Chapter 2) to see this phenomenon in a major political election, with Ralph Nader serving as the potential spoiler candidate. Some have also argued that Gary Johnson and Jill Stein were spoiler candidates in the 2016 U.S. presidential election, and that Hillary Clinton would have won had they not run.

it is also possible that Wayne was just overly optimistic about his sexiness. Perhaps he really thought he had a chance of winning the contest and was honestly giving it his best effort. In either case, Wayne's presence or absence in the contest should have been, for all practical purposes, irrelevant to its outcome. And while removing Wayne's name from the ballots didn't change any of the individual judges' orderings of Dale and Paul, it did change the outcome of the contest.

It's also worth pointing out that this sort of behavior can occur and is particularly troubling in situations where a candidate dies before an election, but not in time to be removed from the ballots. Placed in a slightly different context, the example from Question 4.2 shows that a deceased candidate's very presence on the ballots may alter the outcome of an election. And while it is possible for a candidate to be elected post mortem (in which case certain rules would have to be in place to govern the selection of a replacement), what happens far more often is that the candidate becomes a spoiler—not having a serious chance of winning the election, but affecting the outcome nonetheless.

These observations suggest that we might want to add another item to our list of desirable properties that voting systems should satisfy. The property we'll add captures the substance of our discussion above—specifically, that we want voting systems to be unaffected by the presence or absence of irrelevant candidates. One way to formally express this desire is to say that the societal preference between any two candidates should depend only on the voters' preferences *between those two candidates*, and not on the voters' rankings of any of the other candidates. That way, if society prefers candidate A over candidate B, but then candidate C is removed from the election for some reason, society will still prefer A over B. In other words, the societal preference between A and B should not depend at all on where the ineligible choice C might have appeared on each voter's individual preference ballot. That information ought to be irrelevant, just as C is. We formalize this idea in the following definition.

Definition 4.3. If a voting system has the property that the societal preference between any two candidates depends only on the voters' preferences between those two candidates, then the system is said to satisfy the **independence of irrelevant alternatives** criterion (**IIA** for short).

To state Definition 4.3 another way, with a voting system that satisfies IIA, if some or all of the voters in an election change their preference ballots but no voter changes their preference between two candidates A and B, then the societal preference between A and B must also remain unchanged.

Question 4.4.* Does Black's system satisfy IIA? Why or why not?

Question 4.5. Does the Borda count satisfy IIA? Why or why not?

Question 4.6. Does plurality satisfy IIA? Why or why not?

Question 4.7. Suppose that, in a rematch among the three finalists in the World's Sexiest Man contest, the judges rank Dale, Paul, and Wayne as shown in the preference schedule in Table 4.2.

	Number of Voters		
Rank	5	5	5
1	P	D	W
2	D	W	P
3	W	P	D

TABLE 4.2. World's Sexiest Man rematch

(a) Who would win the contest if the winner was decided by sequential pairwise voting with the agenda D, P, W?

(b) Suppose that, after some private "negotiations," the five judges represented in the rightmost column of Table 4.2 swap the positions of Paul and Dale in their rankings, yielding the new preference schedule shown in Table 4.3. Who would win the contest with this new preference schedule if the winner was again decided by sequential pairwise voting with the agenda D, P, W?

	Number of Voters		
Rank	5	5	5
1	P	D	W
2	D	W	D
3	W	P	P

TABLE 4.3. World's Sexiest Man rematch, revised

(c) What do your answers to parts (a) and (b) allow you to conclude about sequential pairwise voting and IIA? Explain.

Question 4.8. Scoffing at the superficial nature of the World's Sexiest Man contest, the three finalists' wives, Katie, Pam, and Rachel, enter the more civilized "World's Wittiest Woman" contest. Instead of sequential pairwise voting, this contest is to be decided using instant runoff, and the fifteen judges rank Katie, Pam, and Rachel as shown in Table 4.4.

(a) What ranking (societal preference order) of Katie, Pam, and Rachel would be produced from this preference schedule?

(b) In a rematch, the six judges represented in the leftmost column of Table 4.4 swap the positions of Pam and Rachel in their rankings, yielding the new preference schedule shown in Table 4.5. What ranking would be produced from this new schedule?

	Number of Voters			
Rank	6	3	3	3
1	P	K	R	K
2	R	P	P	R
3	K	R	K	P

TABLE 4.4. World's Wittiest Woman rankings

	Number of Voters			
Rank	6	3	3	3
1	R	K	R	K
2	P	P	P	R
3	K	R	K	P

TABLE 4.5. World's Wittiest Woman rematch

(c) What do your answers to parts (a) and (b) allow you to conclude about instant runoff and IIA? Explain.

Question 4.9.* Of the voting systems we've studied so far—plurality, the Borda count, sequential pairwise voting, instant runoff, and Black's system—which satisfy IIA, and which violate IIA?

Actually, the five voting systems listed in Question 4.9 are not the only ones we have studied. We left out dictatorships, imposed rule, and minority rule, which we discussed back in Chapter 1. These systems each had at least one obvious and serious flaw that caused us to rule them out even for elections with just two candidates. But by now you've surely noticed that we're having a hard time finding a voting system that satisfies the very reasonable list of desirable properties we have constructed. In fact, every voting system we've considered has had at least one notable fault that caused us to keep searching for a better system. You may have felt like throwing in the towel a while back. Or perhaps you're still optimistic that a perfect voting system is out there, and we just need to look a bit harder.

Or perhaps you're a pragmatist. You may reasonably argue that Black's system is the best one we've looked at. It satisfies all of our desirable properties except IIA, and none of the other systems we've looked at satisfy IIA either—or do they? Remember, we haven't yet considered whether dictatorships, imposed rule, and minority rule satisfy IIA.

Question 4.10.*

(a) Do dictatorships satisfy IIA? Why or why not?

(b) Does imposed rule satisfy IIA? Why or why not?

(c) Does minority rule satisfy IIA? Why or why not?

So, as it turns out, there *are* voting systems that satisfy IIA. This shouldn't be too surprising, since IIA does seem to be a pretty reasonable criterion. A far more interesting question is this: Are there voting systems that satisfy IIA *and* some or all of the other desirable properties we've mentioned? For example, is there a voting system that is anonymous, neutral, and monotone and also satisfies IIA? As we'll see in the next section, the answer to this question is surprising, disturbing, and very significant.

Arrow's Theorem

In 1951, Kenneth Arrow, who was an economist (like Duncan Black), set out on a quest quite similar to the one in which we have been engaged for the last several chapters. Like us, he wanted to find a voting system that was "fair" according to some reasonably defined standards. And like us, he encountered some fairly major roadblocks along the way. Arrow described his experience as follows:

> *I started out with some examples. I had already discovered that these led to some problems. The next thing that was reasonable was to write down a condition that I could outlaw. Then I constructed another example, another method that seemed to meet that problem, and something else didn't seem very right about it. Then I had to postulate that we have some other property. I found I was having difficulty satisfying all of these properties that I thought were desirable... After having formulated three or four conditions of this kind, I kept on experimenting. And lo and behold,* ***no matter what I did, there was nothing that would satisfy these axioms****.* [emphasis added]

Does this sound familiar to you? It should! More than half a century later, we've been trying to do exactly the same thing as Arrow, and we've been running into exactly the same problems that he did. So how did Arrow resolve his difficulties? Here's more of what he had to say about his experience:

> *So after a few days of this, I began to get the idea that maybe ... there was no voting method that would satisfy all of the conditions that I regarded as rational and reasonable. It was at this point that I set out to prove it. And it actually turned out to be a matter of only a few days work.*[2]

What Arrow probably didn't realize at the time was that those few days of work would help earn him the 1972 Nobel Prize in economic science, and that his "impossibility theorem" would come to be regarded as the single

[2]Both quotes on this page are from an interview that appears in COMAP [**17**].

most important and well-known result in the history of voting theory. And lest you think that a result of this stature would be comprehensible only to experts, take heart—our investigations throughout the last few chapters have prepared us to understand Arrow's Theorem and even to see why it's true. We'll begin by investigating the definitions and conditions that formed the foundation of Arrow's work.

What is a Voting System?

In Arrow's world, a voting system was a rule that assigned a societal preference order to each possible collection of individual preference orders. To use mathematical language, we could say that a voting system is a *function*; we input into the function the preference orders of all the voters in an election, and the function then spits out an overall ranking of the candidates that in some way represents the will of the electorate.

The fact that, like Arrow, we want voting systems to produce well-defined rankings of the candidates (and not just winners) is very important. For instance, recall from the last chapter that we had problems identifying complete rankings of all the candidates in elections conducted using sequential pairwise voting. The main issue that arose at that time was related to a property that mathematicians call *transitivity*.

In the context of elections, the idea behind transitivity is that if society prefers some candidate A over another candidate B, and also prefers B over a third candidate C, then society ought to prefer A over C. If this is true for any combination of three candidates in an election, then we say that the resulting societal preference order is **transitive**. Recall, however, that the exact opposite happened in Condorcet's paradox (see Question 3.4): Society preferred A over B and B over C, but also C over A. In fact, Condorcet's paradox is the classic example of a voting system failing to produce a transitive societal preference order.

Question 4.11.* Suppose X, Y, and Z are the three candidates in an election.

(a) If you know that society prefers X over Z, Z over Y, and X over Y, can you conclude that the resulting societal preference order would be transitive? Explain.

(b) If you know only that society prefers X over Y and Z over X, what would the societal preference between Y and Z have to be in order for the resulting societal preference order to be transitive?

(c) If a fourth candidate entered the election, would your answer to part (a) necessarily be the same? Explain.

Sometimes when societal preference orders fail to satisfy the property of transitivity, we say that the societal preferences represented are *cyclic*. Again, Condorcet's paradox provides a good example of why this wording

is appropriate; if we try to combine the results of each of the pairwise comparisons for Question 3.4, the resulting societal preference order would look something like this:

$$A \succ B \succ C \succ A \succ B \succ C \succ A \succ B \succ C \succ A \succ B \succ C \succ \cdots$$

Recall that in each of the pairwise comparisons that make up this strange societal ranking, the margin of victory was two votes to one. In other words, two thirds of the voters preferred A over B, two thirds preferred B over C, and two thirds preferred C over A.

Question 4.12.* Consider the cyclic societal preferences shown above.

(a) Why might the voters in the election react negatively if A were chosen as the winner of the election? Why might they react negatively if B were chosen as the winner? What about if C were chosen as the winner?

(b) Using your answer to part (a), write a convincing argument for why voting systems that are capable of producing cyclic societal preferences should be avoided.

Given our observations above, from this point forward we will restrict our consideration to voting systems that avoid cyclic societal preferences. That is, we will require the voting systems we consider to produce only transitive societal preference orders. In addition, since we can't expect a voting system to produce something meaningful out of nonsense, we will require the preferences of each of the individual voters to be transitive as well (as has been the case in every example we've considered thus far). With these conditions, we can now formally define a voting system as follows:

Definition 4.13. A **voting system** is a function that receives as input a collection of transitive preference ballots and produces as output a transitive societal preference order.

It's worth noting that nothing in Definition 4.13 rules out ties, either in individual preference ballots or in the societal preference orders produced by a voting system. So, for instance, preference ballots that look like $A \succ B \approx C \succ D$ are perfectly acceptable, as are societal preference orders such as $A \succ B \approx C \approx D$. Thus, with the convention we adopted in Chapter 3, even sequential pairwise voting can be viewed as a voting system in the sense of Definition 4.13.

Arrow's Conditions

Now that we've given a precise definition of what a voting system is, we're ready to move on and state precisely the conditions that Arrow thought every reasonable voting system should satisfy. We'll use the same names that Arrow did in his 1951 book, *Social Choice and Individual Values*.

Condition 1: **Universality.** Voting systems should not place any restrictions other than transitivity on how voters can rank the candidates in an election. Specifically, voting systems should not dictate that some preference orders are acceptable while others are not; *every* possible collection of transitive preference ballots must yield a transitive societal preference order.

Condition 2: **Positive Association of Social and Individual Values.** Voting systems should be monotone.

Condition 3: **Independence of Irrelevant Alternatives.** Voting systems should satisfy IIA.

Condition 4: **Citizen Sovereignty.** Voting systems should not be imposed in any way. That is, there should never be a pair of candidates, say A and B, such that A is always preferred over or tied with B in the resulting societal preference order, regardless of how any of the voters vote.

Condition 5: **Non-dictatorship.** Voting systems should not be dictatorial. That is, there should never be a particular voter—a *dictator*—such that for any pair of candidates A and B, if the dictator prefers A over B, then society will also prefer A over B.

Question 4.14. Which of Arrow's five conditions is most closely related to the property of anonymity as we defined it in Chapter 2? Which is most closely related to neutrality?

Note that all but one of Arrow's conditions are quite similar to the desirable properties we've studied already. The one that we haven't yet stated explicitly—but have assumed implicitly—is the first; it merely says that a reasonable voting system ought to let voters vote however they want. After all, we might really like majority rule, but majority rule with the added condition that everyone must vote for candidate A is anything but fair.

With that said, it's important to note that a voting system can violate the condition of universality without explicitly placing any restrictions on the ballots voters are allowed to cast. How can this be? Well, recall that in Definition 4.13 we said that a voting system must always output a *transitive* societal preference order. As we saw earlier, however, some potential voting systems, such as the one that produced Condorcet's paradox, naturally yield cyclic societal preferences when faced with certain collections of individual preferences. To deal with such systems, we have two choices: Either we can say that the system in question is not really a voting system according to Definition 4.13 (since it is capable of producing cyclic societal preferences), or we can say that it *is* a voting system but that it can only produce a transitive societal preference order for certain collections of individual preferences. With this latter option, the system would violate universality only because its set of potential inputs would have to be restricted in order to

guarantee transitive societal preferences. The next question illustrates how such a restriction could work.

Question 4.15.

(a) Suppose that in a three-candidate election between candidates A, B, and C, the only individual preference ballots allowed are $A \succ B \succ C$ and $B \succ A \succ C$. Suppose also that the societal preference order for the election is to be formed by simply combining the results of each of the three possible pairwise comparisons (as we did earlier to produce Condorcet's paradox). Explain why, with only these two ballots allowed, it is impossible for the resulting societal preference order to be cyclic.

(b) Suppose that a third ballot, $C \succ B \succ A$, is also allowed. Could the societal preference order be cyclic in this case? Why or why not?

The Punchline

And now, the moment we've been waiting for—a theorem that is both beautiful from a mathematical standpoint and at the same time devastating to our search for the perfect voting system.

Arrow's Theorem. *For an election with more than two candidates, it is impossible for a voting system to satisfy all five of Arrow's conditions.*

The precise wording of Arrow's Theorem is extremely important. The theorem does *not* tell us that mathematicians and social scientists just haven't yet found a voting system that satisfies all five of Arrow's conditions (but perhaps might someday). It also does not tell us that it will be *really difficult* to find such a system. Instead, what Arrow's Theorem says is that it is *impossible* for us or anyone else to do so. Try as we might, we will never find a voting system for an election with more than two candidates that satisfies all five of Arrow's very basic and desirable conditions. In other words, every voting system we could ever discover or invent would necessarily have to violate at least one of these five conditions. We can obviously find many systems that satisfy some of Arrow's conditions—and we have done so already—but we will never be able to find a voting system that satisfies *all* of them.

Question 4.16. Which of Arrow's five conditions do you think is the least important for a voting system to satisfy? Give a convincing argument to justify your answer.

Question 4.17.

(a) In your answer to Question 4.6, you probably gave a specific example to show that plurality does not satisfy IIA. Now use Arrow's Theorem (without a specific example) to give another explanation for why this is true.

(b) Could you use Arrow's Theorem, as you did in part (a), to show that instant runoff does not satisfy IIA? Why or why not?

Question 4.18.

(a) Explain how you know that any voting system that satisfies IIA is anonymous, neutral, and monotone must necessarily satisfy Arrow's conditions 2–5.

(b) Explain why the statement in part (a) implies that any voting system that satisfies IIA and is anonymous, neutral, and monotone must necessarily violate universality.

Arrow himself did not think it was at all unreasonable to require that voting systems satisfy universality; nor could he conceive of a reasonable voting system that would ever violate monotonicity or IIA. Thus, he interpreted his theorem as saying that "the only methods of passing from individual tastes to social preferences which will be satisfactory and which will be defined for a wide range of sets of individual preferences are either imposed or dictatorial."

Others have interpreted Arrow's Theorem differently, and there has actually been a fair amount of debate about what Arrow's Theorem really means and how it should be interpreted in light of the search for a voting system that is truly fair. In the next chapter, we'll consider some of these other interpretations and investigate some potential resolutions to the difficulties revealed by Arrow's work. But before doing so, let's take a quick look at an important and useful variation of Arrow's Theorem.

Pareto's Unanimity Condition

Arrow's Theorem is a surprisingly strong result, but it can actually be made even stronger. Without altering the truth of the theorem, Arrow's conditions 2 and 4 (monotonicity and citizen sovereignty) can be replaced with the following *unanimity* condition, which is sometimes also referred to as the *Pareto condition*, in honor of Vilfredo Pareto (yes, it rhymes!), an Italian economist and political activist who lived in the late 1800s and early 1900s.

The Pareto Condition: **Unanimity.** If there is a pair of candidates in an election, say A and B, such that *every* voter in the election prefers A over B, then A should be ranked higher than B in the resulting societal preference order.

Question 4.19.* Suppose that candidate A is selected as the winner of an election. For each of the following scenarios, decide, if possible from the information given, whether the voting system used in the election satisfies or violates unanimity. Clearly explain each of your answers.

(a) Candidate A receives no first-place votes; that is, every voter in the election prefers at least one other candidate over A.

(b) There is a candidate in the election, say B, such that every voter in the election prefers B over A.

Question 4.20.*

(a) Does plurality satisfy unanimity? Why or why not?

(b) Does the Borda count satisfy unanimity? Why or why not?

(c) Does instant runoff satisfy unanimity? Why or why not?

Unanimity, like some of the other desirable properties we've considered, seems at first glance to be very reasonable. In fact, unanimity is such a natural and obvious condition that we might expect it to be satisfied by any voting system we could think of. But, as we saw in Question 4.20, this is not the case. In fact, the stronger form of Arrow's Theorem stated below tells us that any voting system that satisfies unanimity will necessarily violate at least one of Arrow's other conditions.

Arrow's Theorem (Strong Form). *For an election with more than two candidates, it is impossible for a voting system to satisfy unanimity and Arrow's conditions* 1*,* 3*, and* 5*.*

Stated differently, the strong form of Arrow's Theorem says that every voting system that does not dictate the preferences of voters and is not equivalent to a dictatorship must violate either IIA or unanimity (or both). Moreover, some violations of unanimity—like the one in the next question—can be particularly grievous.

Question 4.21. Consider the preference schedule shown in Table 4.6 for an election with four candidates.

	Number of Voters		
Rank	1	1	1
1	A	B	C
2	B	C	D
3	C	D	A
4	D	A	B

TABLE 4.6. Sequential pairwise voting and unanimity

(a) Find an agenda for which candidate D would win the election under sequential pairwise voting.

(b) Clearly explain why your answer to part (a) shows that sequential pairwise voting does not satisfy unanimity.

(c) How is the violation of unanimity that you observed in this question worse in some sense than those you observed in Question 4.20? Explain.

Concluding Remarks

This chapter dealt a seemingly devastating blow to our search for the perfect voting system. It should have also raised a number of questions, such as:

- How can we prove that something is impossible?
- Are Arrow's conditions as reasonable as they seem at first glance?
- If no voting systems are perfect, which ones are best?
- What are some ways of resolving the difficulties illuminated by Arrow's Theorem?

We'll consider these and other questions in the next chapter.

Questions for Further Study

Question 4.22. In this and the previous two chapters, we've considered five voting systems for elections with more than two candidates: plurality, the Borda count, sequential pairwise voting, instant runoff, and Black's system. How do these systems compare with each other when applied to elections with only two candidates? Explain.

Question 4.23. Is Arrow's Theorem true for elections with only two candidates? If so, explain why. Otherwise, give an example of a voting system for an election with two candidates that satisfies all five of Arrow's conditions.

Question 4.24. Of all the desirable properties for voting systems that we've discussed so far, which are satisfied by dictatorships? By imposed rule? By minority rule?

Question 4.25. For each part below, find or invent a voting system for an election with more than two candidates that satisfies all of the three properties listed.

(a) universality, IIA, unanimity

(b) universality, IIA, non-dictatorship

(c) universality, unanimity, non-dictatorship

(d) IIA, unanimity, non-dictatorship

Question 4.26. Find an article about Arrow's impossibility theorem in a popular media source. Write a summary and critique of the article based on what you have learned in this chapter.

Question 4.27. Write a short biography of Kenneth Arrow, including his most important contributions both inside and outside of voting theory.

Question 4.28. Write a short biography of Duncan Black, including his most important contributions both inside and outside of voting theory.

Question 4.29. Write a short biography of Vilfredo Pareto, including his most important contributions both inside and outside of voting theory, some information about his political views, and some of the personal problems he faced.

Question 4.30. Black's voting system is an example of a *Condorcet completion system*, meaning that it elects a Condorcet winner if one exists but reverts to some other voting system if one doesn't exist. Research another Condorcet completion system of your own choosing, and write a detailed summary of your findings. Include in your summary a complete evaluation of the system you chose using the criteria we have discussed in this and previous chapters.

Question 4.31. Suppose we redefine a voting system to be a rule that assigns to each possible collection of transitive preference ballots a winning candidate or collection of winning candidates (as opposed to assigning a transitive societal preference order). With this new definition, would Arrow's Theorem still apply? Give a convincing argument to justify your answer. (Hint: You may want to look back at our discussion of sequential pairwise voting and societal preference orders beginning on page 39.)

Question 4.32. Find out how judging is conducted for figure skating competitions at the Winter Olympic Games, and write a summary of your findings. Include in your summary the actual final ranking and numerical figures from the competition at a recent Olympics.

Question 4.33. Many of the ranking systems used to judge figure skating competitions do not satisfy IIA. Find a magazine, newspaper, or web site that includes an actual example illustrating such a violation. Write a detailed summary of your findings, including the competition in which the incident occurred, the final outcome of the competition, a complete description of the ranking system used by the judges (you may refer to your answer to Question 4.32 if it is the same), and an explanation for how you know that a violation of IIA occurred.

Question 4.34. Consider the following voting system for an election with more than two candidates: Each possible pair of candidates is compared in a head-to-head contest, with one point awarded to the winner, or one-half point awarded to each in the case of a tie. After all the head-to-head comparisons have been completed, the candidate who has been awarded the largest total number of points (or candidates in the case of a tie) is declared the overall winner of the election. This system is often called the *method of pairwise comparisons*.

(a) If the method of pairwise comparisons were used for the PU mathematics chair election from Question 3.17, who would the winner be?

(b) Describe the natural way to construct societal preference orders using the method of pairwise comparisons. Then find the societal

preference order produced by the method of pairwise comparisons for the CVAAB presidential election from Question 2.8.

(c) How many head-to-head comparisons would be required for the method of pairwise comparisons in an election with four candidates? What if there were five candidates? What about n candidates (where n represents some arbitrary whole number)?

(d) Describe some of the pros and cons of the method of pairwise comparisons. How does it compare to the other systems we have discussed (plurality, the Borda count, sequential pairwise voting, instant runoff, and Black's system)?

(e) Which of the criteria we have considered (anonymity, neutrality, monotonicity, the majority criterion, the CWC, the CLC, IIA, and unanimity) are satisfied by the method of pairwise comparisons? Which are violated? Explain your answers in detail, and give convincing arguments to justify your claims.

(f) Find a magazine, newspaper, or web site that describes an example where the method of pairwise comparisons was used to arrive at some type of decision or ranking. Write a summary of your findings, including the name of your source and the outcome of the example.

Question 4.35. Research the voting system used by the hit TV show *American Idol*, and write a detailed summary of your findings. Include in your summary a comparison of this voting system with the other systems we have investigated, an analysis of this voting system according to the fairness criteria we have developed, and a discussion of some of the controversy surrounding the voting system.

Question 4.36. Arrow's Theorem was the first of several impossibility theorems to be proved during the second half of the twentieth century. Other well-known results from this time period include Sen's Theorem (also known as Sen's paradox) and the Duggan-Schwartz Theorem. Investigate each of these theorems, and write a detailed summary of your findings. Include in your summary a description and critique of the conditions used in each theorem, how these conditions are related to Arrow's conditions, and a brief biography of the individual(s) for whom each theorem is named.

Question 4.37. In Chapter 3 (Question 3.14), we argued that sequential pairwise voting does not satisfy neutrality. Does sequential pairwise voting satisfy citizen sovereignty? Give a convincing argument or example to justify your answer.

Question 4.38. Find a voting system that satisfies Arrow's non-dictatorship condition but is not anonymous, or explain why no such system exists.

Question 4.39. Research each of the following voting systems, and write a detailed summary of your findings. For each system, include in your

summary a brief description of the system, an example to illustrate it, and a discussion of which fairness criteria are satisfied by the system.

(a) The Kemeny-Young method

(b) The minimax method (also called Simpson's method)

(c) The Schulze method (also called Schwartz sequential dropping)

(d) The ranked pairs method (also called Tideman's method)

What do all of these systems have in common?

Question 4.40. Research each of the following criteria for evaluating voting systems, and relate them to the other criteria we have discussed so far. Which of these criteria seem most important to you?

(a) The Smith criterion

(b) Reinforcement

(c) Reversal symmetry

(d) Independence of clones

(e) Consistency

(f) Participation

Answers to Starred Questions

4.2. (a) Black's system would revert to the Borda count. Dale would be selected as the winner, and the resulting societal preference order would be $D \succ P \succ W$.

(b) Since Wayne is ranked last in the societal preference order in part (a), his exclusion from the contest shouldn't change the outcome.

(c) With Wayne excluded from the race, Black's system would select Paul as the winner.

(d) It is indeed strange that excluding the last-place candidate from the ranking in part (a) changed the winner of the contest from Dale to Paul.

4.4. Black's system does not satisfy IIA. The example from Question 4.2 shows this.

4.9. None of the systems listed satisfy IIA.

4.10. Both dictatorships and imposed rule satisfy IIA.

4.11. (a) The pairwise preferences given can yield only one possible societal preference order, $X \succ Z \succ Y$, which is transitive.

(b) In order for the resulting societal preference order to be transitive, it would have to be the case that $Z \succ Y$.

(c) With a fourth candidate included in the election, the information given in part (a) would not be sufficient to determine whether the resulting societal preference order would have to be transitive. We would also need to know how society viewed the new candidate in comparison to each of X, Y, and Z.

4.12. (a) If A were chosen as the winner, two thirds of the voters in the election would prefer C. But if B were chosen as the winner, two thirds of the voters would prefer A. And if C were chosen, two thirds of the voters would prefer B.

4.19. (a) Knowing that A wins without any first-place votes does not allow us to conclude that the voting system violates unanimity.

(b) Since every voter prefers B over A, if the system satisfied unanimity, then B would have to be ranked higher than A in the resulting societal preference order. But this is impossible, of course, if A wins the election. Therefore, in this case, the system violates unanimity.

4.20. (a) Plurality *almost* satisfies unanimity, but not quite. Consider an election with three candidates in which every voter has the preference order $A \succ B \succ C$. In this case, every voter would prefer B over C, but B and C would be tied in the resulting societal preference order since neither would receive any first-place votes.

(b) The Borda count satisfies unanimity. (Can you explain why?)

(c) Instant runoff violates unanimity for the same reason that plurality does.

Chapter 5

Explaining the Impossible

No one pretends that democracy is perfect or all-wise. Indeed it has been said that democracy is the worst form of government except for all those other forms that have been tried from time to time.

– Winston Churchill

Focus Questions

In this chapter, we'll explore the following questions:

- What is the basic idea behind the proof of Arrow's Theorem?
- Can Pareto's unanimity condition be weakened to solve the problems revealed by Arrow's Theorem?
- What is approval voting? Does it solve any of the problems revealed by Arrow's Theorem?
- What is the intensity of binary independence criterion? How is it related to Arrow's Theorem?

Warmup 5.1. Consider the following mathematical claim:

> *It is impossible for a whole number to be divisible by* 2*,* 11*, and* 23 *and not be greater than* 500*.*

Is this claim true or false? Give a convincing argument or example to justify your answer.

Suppose we wanted to prove that the claim from Warmup 5.1 is true. How could we do it? One method would be to simply check all of the whole numbers one by one and verify that none of them are divisible by 2, 11, and 23 and not greater than 500. However, this would take quite a long time, wouldn't it? Actually, since there are infinitely many whole numbers, the truth is we'd never be able to check them all.

Of course, we could reduce our work quite a bit if we only considered the whole numbers that are not greater than 500. Then we'd just have to

show that none of these numbers are divisible by 2, 11, and 23. But even this seems like an awful lot of work.

Fortunately, there's a much better way to prove that the claim from Warmup 5.1 is true. What if, instead of considering numbers one by one, we constructed some sort of logical argument to establish the truth of the claim? For instance, we might say something like this:

> *The numbers* 2, 11, *and* 23 *are all prime numbers. Hence, any whole number divisible by* 2, 11, *and* 23 *must be at least as big as* $2 \times 11 \times 23 = 506$. *So it is impossible for a whole number to be divisible by* 2, 11, *and* 23 *and not be greater than* 500.

At this point, you might be wondering what all of this has to do with voting. Well, as it turns out, the same strategy we just used to prove the claim from Warmup 5.1 can also be used to prove Arrow's Theorem—and we'll do so in this chapter. Our goal will be to see *why* Arrow's Theorem is true and to consider some potential options for resolving the problems that Arrow first brought to light.

Proving Arrow's Theorem

We'll begin by walking step by step through a proof of Arrow's Theorem.[1] First, you should be advised that Arrow's Theorem is a significant result, and proving it will require a fair amount of effort and concentration. Nevertheless, we'll be able to tackle and understand the proof if we just take it one step at a time. When we're done, you will have joined a select group of people who know not only the meaning of one of the most important theorems in social choice theory, but also why the theorem is true.

Before we take off into the proof, we need to mention one more piece of notation that we will be using along the way. Recall that we have used the symbol $\succ$ to represent a preference between two candidates in an election and $\approx$ to represent a tie. Sometimes in this chapter we will have reason to say that a candidate A is either preferred over *or* tied with another candidate B. We will represent this type of relation by writing $A \succsim B$. Note again the analogy to a common symbol ($\geq$) that we use to compare numbers.

Now on to Arrow's Theorem. As strange as it may seem, it is actually easier to prove the strong form of the theorem that we stated later in Chapter 4 (on page 64) than it is to prove the original version that we stated earlier in that chapter (on page 62). So we'll prove the strong form of Arrow's Theorem first, and then look more carefully at why the strong form implies the original form. To make our proof strategy more clear, we'll begin by restating the strong form of the theorem in a slightly different way from how we first stated it on page 64.

[1] The strategy we will use to prove Arrow's Theorem was adapted from a paper by Geanakoplos [**22**].

Arrow's Theorem (Strong Form). *For an election with more than two candidates, it is impossible for a voting system that satisfies universality, IIA, and unanimity to not be a dictatorship.*

Question 5.2. Explain why the strong form of Arrow's Theorem as stated above is equivalent to how we first stated it on page 64.

If you look carefully, you'll notice that the strong form of Arrow's Theorem, as it is stated above, is similar in style to the mathematical claim we considered in Warmup 5.1. And, just as we thought about doing there, we could try to prove Arrow's Theorem by using a brute-force approach—that is, by checking every possible voting system and verifying that none of them satisfies all four conditions. However, it's hard to imagine how we would go about checking *every* possible voting system. In fact, it's not even clear that we could *identify* all of the possible voting systems, let alone investigate the properties of each one.

A much better approach would be to try to do exactly what we did in our answer to Warmup 5.1. There, we simply assumed that three of the conditions were true (divisible by 2, 11, and 23) and then explained why the fourth condition (not greater than 500) could not also be true. This is where our revised wording comes in particularly handy. It tells us that we can begin our proof by assuming that there is some voting system for an election with more than two candidates that satisfies universality, IIA, and unanimity. To complete the proof, we'll then just need to explain why this voting system must be equivalent to a dictatorship. That is, we'll need to show that there is some voter v in the system such that for any two candidates A and B, if v prefers A over B, society will also prefer A over B.

Incidentally, it's worth pointing out the similarities between the last step in our proof strategy above and what we did in Chapter 1 when we proved Theorem 1.22. In that theorem, we assumed that we had a voting system that satisfied anonymity, neutrality, and monotonicity, and we needed to show that this voting system was equivalent to a quota system. To do this, we first constructed a process through which we found a potential value for the quota. We then showed that this potential quota actually worked the way that the quota in a quota system is supposed to work.

Our strategy here will be similar. We'll first construct a process through which we will find a potential dictator for our hypothetical voting system. We'll then show that this potential dictator actually *is* a dictator according to our definition.

In order to do all of this, we'll first need to consider a lemma[2] that will help us along the way. Although right now you may not see exactly how we'll use this lemma, be assured that it will play a crucial role in our proof of Arrow's Theorem.

[22]The word *lemma* means *helping result*. In mathematics, a lemma is typically a result whose main use is to help establish the truth of another more important result.

Lemma 5.3. *Assume that for an election with more than two candidates, a voting system V satisfies universality, IIA, and unanimity. Suppose that B is some candidate in the election, and that every voter ranks B in either first place or last place (without ties) in their individual preference order. Then the societal preference order produced by V must also rank B either first or last—even if, for example, half of the voters rank B first and the other half rank B last.*

Note that, throughout this chapter, when we say that a candidate is ranked first or last in an individual or societal preference order, we rule out the possibility that they are tied with another candidate. You'll want to keep this in mind as you answer the questions that build up our proof of Lemma 5.3 and Arrow's Theorem.

In order to see why Lemma 5.3 is true, let's begin by assuming that every voter in the election does in fact rank B in either first or last place. We'll make no other assumptions about the voters' preferences.

Question 5.4.* Suppose that the societal preference order produced by V does not rank B either first or last. Explain why it must then be the case that for some other candidates A and C, $A \succsim B$ and $B \succsim C$.

Question 5.5. Given that $A \succsim B$ and $B \succsim C$, what does the transitivity of the societal preference order allow you to conclude about the societal preference between A and C?

Question 5.6.* Now suppose that every voter changes their individual preference order by moving C above A, with no other changes. Will these changes have any effect on the resulting societal preference between A and B or between B and C? Explain. (Hint: Don't forget that every voter ranks B in either first or last place, without ties, in their individual preference order.)

Question 5.7. In Question 5.6, we assumed a change in preferences that resulted in every voter ranking C above A. Given this change, what does unanimity allow you to conclude about the resulting societal preference between A and C?

Question 5.8.* Explain how your answers to Questions 5.5 through 5.7 give rise to a contradiction. What does this contradiction allow you to conclude about the truth of Lemma 5.3? Thoroughly explain your answer.

With the truth of Lemma 5.3 established, let's now proceed to find our potential dictator. From now through Question 5.16, we'll suppose that we have an election with more than two candidates and a voting system V that satisfies universality, IIA, and unanimity (so that Lemma 5.3 applies). For convenience, we'll name the voters in the election $v_1, v_2, v_3, \ldots, v_n$ (where n represents the total number of voters). And, for reasons that will become

clear later, we'll begin by considering the special case in which B is ranked last (without ties) by all of the voters in the system, as shown in Table 5.1.

	Voters			
Rank	v_1	v_2	$\cdots$	v_n
First	?	?	$\cdots$	?
$\vdots$	$\vdots$	$\vdots$		$\vdots$
Last	B	B	$\cdots$	B

TABLE 5.1. Candidate B ranked last unanimously

Question 5.9. Based on the fact that B is ranked last by every voter, what can you conclude about the position of B in the societal preference order produced by V? What property allows you to conclude this?

Question 5.10.*

(a) Suppose that *all* of the voters move B from last place to first place in their individual preference orders. How would the resulting societal preference order change, and why would this change occur?

(b) Suppose that only *some* of the voters move B from last place to first place in their individual preference orders. What possible changes could occur in the resulting societal preference order? (Hint: Don't forget about Lemma 5.3!)

(c) Suppose that, one by one, starting with v_1 and proceeding in order, each voter moves B from last place to first place in their individual preference order. Explain why there must be some voter, say v_j, for which this move first causes a corresponding change to occur in the societal preference order.

The voter that we labeled v_j in part (c) of Question 5.10 is a special voter in the following sense: Even if all of the voters before v_j moved B from last place to first place in their individual preference orders (as shown in Table 5.2), there would still be no change to B's position in the resulting societal preference order. However, as soon as v_j makes the same change (as shown in Table 5.3), suddenly B would move from last to first in the resulting societal preference order. Because of this, we could call v_j a *pivotal* voter.

As it turns out, v_j is also a dictator. We'll establish this fact in two steps: First, we'll show that for any pair of candidates that does not include B, say A and C, if v_j prefers A over C, then society will also prefer A over C. We'll then show that the same condition holds for any pair of candidates that does include B.

	Voters							
Rank	v_1	v_2	$\cdots$	v_{j-1}	v_j	v_{j+1}	$\cdots$	v_n
First	B	B	$\cdots$	B	?	?	$\cdots$	?
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
Last	?	?	$\cdots$	?	B	B	$\cdots$	B

TABLE 5.2. Society ranks B last

	Voters							
Rank	v_1	v_2	$\cdots$	v_{j-1}	v_j	v_{j+1}	$\cdots$	v_n
First	B	B	$\cdots$	B	B	?	$\cdots$	?
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
Last	?	?	$\cdots$	?	?	B	$\cdots$	B

TABLE 5.3. Society ranks B first

For the first step, let A and C represent any two candidates other than B. In addition, assume that v_j prefers A over C. We now want to be able to conclude that A is preferred over C in the resulting societal preference order. Because we need to show that A is preferred over C regardless of the preferences of any of the other voters besides v_j, we won't make any assumptions about the preferences of these other voters. The only thing we'll assume is that v_j prefers A over C. For convenience, we'll call the corresponding preference schedule S.

Question 5.11.* Would any of the following changes to S affect the resulting societal preference between A and C? Explain your answer in each case.

(a) Voter v_j moves B between A and C in their individual preference order.

(b) Each of the voters $v_1, v_2, \ldots, v_{j-1}$ (all of the voters listed before v_j) moves B to first place in their individual preference order.

(c) Each of the voters $v_{j+1}, v_{j+2}, \ldots, v_n$ (all of the voters listed after v_j) moves B to last place in their individual preference order.

Question 5.12. Suppose all three changes listed in Question 5.11 are made to S, and call the resulting preference schedule S'.

(a) Explain why each voter's relative ranking of just A and B in S' is the same as their relative ranking of just A and B in the preferences shown in Table 5.2.

(b) Use your answer to part (a) to explain why, given the preference schedule S', A would be preferred over B in the resulting societal preference order.

(c) Use the same reasoning that you used in parts (a) and (b) to explain why, given S', B would be preferred over C in the resulting societal preference order.

(d) Now use your answers to parts (b) and (c) to explain why, given S', A would be preferred over C in the resulting societal preference order.

Question 5.13.* What do your answers to Questions 5.11 and 5.12 allow you to conclude about the societal preference order that would be produced by V given the preference schedule S (instead of S')? Explain.

Question 5.14.* Explain how your answer to Question 5.13 allows you to conclude that whenever v_j prefers A over C, society will also prefer A over C (regardless of the preferences of the other voters in the election).

We're actually very close to being done now. Recall that we are trying to show that v_j is a dictator. What we have shown already is that v_j controls the societal preference between any pair of candidates that does not include B. We must now explain why v_j also controls the societal preference between any pair of candidates that does include B.

Here's where we need to be a little sneaky. Remember that we started just before Question 5.9 by assuming B was ranked in last place by every voter. This allowed us to identify a potential dictator, v_j, who controlled the societal preference between any pair of candidates that did not include B. Had we instead started the entire process by assuming that some other candidate, say A, was ranked last by every voter, we would have ended up with another potential dictator, say v_i, who would have controlled the societal preference between any pair of candidates that did not include A.

Question 5.15. Let C be any candidate other than A or B.

(a) Looking back at Question 5.10, clearly explain how v_j could possibly affect the societal preference between B and C. (Hint: Recall that v_j was chosen to be *pivotal* in some sense. You may want to try considering the preferences shown in Tables 5.2 and 5.3 and the resulting societal preference orders.)

(b) What does your answer to part (a) allow you to conclude about the relationship between v_i and v_j? (Hint: Recall that v_i *completely* controls the societal preference between any pair of candidates that does not include A.)

Question 5.16.*

(a) Considering your answer to Question 5.15, is the following statement true or false? Briefly explain how you know.

There exists a single voter v^* that satisfies all three of the following properties:

- v^* completely controls the societal preference between any pair of candidates that does not include A.
- v^* completely controls the societal preference between any pair of candidates that does not include B.
- v^* completely controls the societal preference between any pair of candidates that does not include C.

(b) If there were a voter v^* that satisfied all three of the properties from part (a), what could you conclude about v^*?

Question 5.17. Summarize what you have learned so far in this section by writing a detailed outline of how one could go about proving the strong form of Arrow's Theorem.

Before we can officially put Arrow's Theorem to bed, we still have one final detail to think about. Recall that we obtained the strong form of Arrow's Theorem by replacing Arrow's conditions 2 and 4 (monotonicity and citizen sovereignty) with the Pareto condition (unanimity). The following lemma is what makes this replacement possible.

Lemma 5.18. *If a voting system satisfies monotonicity, IIA, and citizen sovereignty, then it also satisfies unanimity.*

Question 5.19. Explain why Lemma 5.18, along with the strong form of Arrow's Theorem, implies the original form of Arrow's Theorem that we stated on page 62.

We've proved a few "if-then" statements now, so you should be getting the hang of it. As you may have noticed, a good first step in any proof is to identify what we can assume and what we need to show. Let's see if you can figure out these two components for the proof of Lemma 5.18.

Question 5.20.* To prove Lemma 5.18, what should we assume? What should we try to show?

So let's assume that we have a voting system V that satisfies exactly the properties you identified in Question 5.20. We must now show that if every voter in the system prefers A over B, then it will also be the case that A is preferred over B in the societal preference order produced by V. We'll start by assuming that we have some arbitrary preference schedule, say S, in which every voter prefers A over B. We'll then attempt to explain why, given S, A would be preferred over B in the resulting societal preference order. To do this, we'll need to consider two other preference schedules that are related to S. For the first, let S' denote any preference schedule for which A would be preferred over B in the resulting societal preference order.

Question 5.21. Which property that we assumed about V allows us to conclude that such a preference schedule S' actually exists?

For the second, by moving only A in the individual preference orders that make up S', we'll create a new preference schedule in which every voter prefers A over B. Call this new preference schedule S''.

Question 5.22.

(a) With regard only to individual preferences between just A and B, how do the preference schedules S' and S'' differ?

(b) What does your answer to part (a) allow you to conclude about the societal preference between A and B that would be produced by V given S''? Which property that you assumed about V allows you to conclude this?

(c) With regard only to individual preferences between just A and B, how do the preference schedules S and S'' differ?

(d) What does your answer to part (c) allow you to conclude about the societal preference between A and B that would be produced by V given S? Which property that you assumed about V allows you to conclude this?

Question 5.23.* What does your answer to Question 5.22 allow you to conclude about the voting system V and the property of unanimity? Does this finish the proof of Lemma 5.18? Explain.

Potential Solutions

Now that we've seen why both forms of Arrow's Theorem are true, let's look at a few different ways in which we might be able to resolve the problems revealed by Arrow's work.

Weakening the Pareto Condition

As we saw in Chapter 4, unanimity (the Pareto condition) is a fairly strong property. For one thing, it rules out the possibility of two candidates ending up tied in a societal preference order when one of the candidates is unanimously preferred over the other. But, as we've seen, this is exactly the kind of behavior that can occur with plurality or instant runoff when several candidates end up tied with zero first-place votes. Fortunately, Pareto's original unanimity condition can be modified very slightly to allow for such ties.

The Modified Pareto Condition. If there is a pair of candidates in an election, say A and B, such that every voter in the election prefers A over

B, then B should not be ranked higher than A in the resulting societal preference order.

Question 5.24.* Does plurality satisfy the modified Pareto condition? What about instant runoff? Sequential pairwise voting? Give a convincing argument to justify each of your answers.

As it turns out, there are voting systems that satisfy Arrow's conditions 1, 3, and 5, as well as the modified Pareto condition.

Question 5.25. Let V be the voting system in which all candidates are tied in the resulting societal preference order, regardless of the ballots cast. Explain why V satisfies Arrow's conditions 1, 3, and 5 and the modified Pareto condition.

Unfortunately, the voting system from Question 5.25 is not a particularly interesting or useful one. Moreover, thanks to Stanford economist Robert Wilson, we know that there is not much hope of finding a more useful system that satisfies the same conditions. In fact, if we also require neutrality, Wilson's Theorem [**57**] implies that any system that satisfies universality, IIA, and the modified Pareto condition must be a dictatorship or an *inverse* dictatorship (where the societal preference is always the *opposite* of what the dictator wants), or always result in a tie among all of the candidates (as in Question 5.25).

Ballot Restrictions and Approval Voting

In the 1970s, several political analysts independently proposed a new method for deciding the winner of an election with more than two candidates. This method, now commonly referred to as **approval voting**, works as follows:

- Each voter votes to either *approve* or *disapprove* of each candidate in the election. Voters can approve of as many candidates as they wish.
- The societal preference order is determined by the number of approval votes each candidate receives, starting with the candidate who receives the most approval votes and ending with the candidate who receives the fewest (with ties permitted if candidates receive identical numbers of approval votes).

Question 5.26.* Three friends, Peter, James, and John, are trying to decide who among them is the greatest. To do so, they ask nine of their friends to cast approval ballots. The results are shown in Table 5.4, with ✓ indicating a vote of approval.

(a) Under approval voting, who would be declared the greatest? What societal preference order would be produced?

(b) Do you think that the outcome of the election under approval voting accurately reflects the will of the voters? Why or why not?

	Number of Voters		
Candidate	4	3	2
Peter	✓		✓
James		✓	✓
John			

TABLE 5.4. Approval voting

Approval voting has been adopted by a number of scientific and technical organizations, such as the American Mathematical Society, the Institute for Operations Research and Management Science, the American Statistical Association, and the Institute of Electrical and Electronics Engineers. Each of these organizations uses approval voting to elect officers and make other important decisions. Approval voting is also used to elect the Secretary-General of the United Nations, as well as new members of the National Academy of Sciences. It is also used for internal elections within political parties in some states.

Many proponents of approval voting have argued that since the method avoids using ranked ballots, Arrow's Theorem does not apply. At first glance, this conclusion seems entirely logical. In reality, however, the situation is a bit more complex. The real question is this: Can approval voting be viewed as a voting system in the sense of Definition 4.13? That is, can approval voting be viewed as a function that receives as input a collection of transitive preference ballots and produces as output a transitive societal preference order? If so, then Arrow's Theorem still applies. If not, then we may have found the perfect voting system after all. Let's examine this issue a little more closely.

Question 5.27.* Consider the two voters represented in Table 5.4 who approved of both Peter and James. Which of the following individual preference orders could be consistent with these two voters' approval ballots?

(a) Peter $\succ$ James $\succ$ John

(b) James $\succ$ Peter $\succ$ John

(c) Peter $\approx$ James $\succ$ John

(d) James $\approx$ Peter $\approx$ John

(e) Peter $\succ$ James $\approx$ John

Question 5.27 suggests that even though approval voting requires a different type of ballot than the voting systems we've considered previously, the underlying preferences of the voters can in fact be viewed in the same way. Admittedly, the correspondence between approval ballots and preference orders is somewhat loose, since each possible approval ballot will likely

be consistent with many different preference orders (and, likewise, each possible preference order will likely be consistent with many different approval ballots). However, there are some natural conventions we can adopt to help us make the translation between the two.

Let's look back at Question 5.27. Which of the preference orders in that question do you think *best* represents the approval ballots of the two voters who approved of both Peter and James? It would be reasonable to argue that the one from part (c) (Peter $\approx$ James $\succ$ John) is the best choice because it does not specify any sort of ranking between Peter and James. It accurately reflects the information given in the voters' approval ballots and avoids making any additional assumptions about the relative rankings of the candidates, other than those that can be directly inferred from the information supplied on the approval ballots. These observations suggest the following way to formally define approval voting.

Definition 5.28. The voting system known as **approval voting** is characterized by the following two conditions:

- The system accepts as input only those preference orders in which the symbol $\succ$ appears exactly once. In other words, the only preference orders allowed are those that have one or more candidates tied for first place, followed by all of the remaining candidates tied for last place.
- The societal preference order is determined by the number of first-place votes received, starting with the candidate who receives the most and ending with the candidate who receives the fewest (with ties permitted if candidates receive identical numbers of first-place votes).

Incidentally, we will refer to the first-place votes from Definition 5.28 as *approval votes*, a convention that is completely consistent with our more intuitive understanding of approval voting.

Question 5.29.*

(a) Is approval voting, as described in Definition 5.28, a voting system according to Definition 4.13?

(b) By its very definition, approval voting violates one of the important fairness criteria that we have discussed. Which criterion does it violate, and is this violation acceptable or unacceptable in your opinion? Clearly explain your answers.

We can see from Question 5.29 that approval voting—by definition—violates one of Arrow's conditions. But what about Arrow's other conditions, such as the elusive IIA?

Question 5.30.* Suppose an election is held using approval voting, but, due to voter irregularities, a revote is necessary. Suppose also that in this

revote, some voters change their ballots, but never in a way that affects their individual preferences between just candidates A and B.

(a) Explain why, in the revote, the difference in the number of approval votes received by A and B will be exactly the same as it was in the original election.

(b) What does your answer to part (a) allow you to conclude about approval voting and IIA?

Question 5.31.*

(a) Is approval voting anonymous? Neutral? Monotone? Clearly explain your answers.

(b) Does approval voting satisfy the Pareto condition? Why or why not?

Question 5.32. Suppose that approval voting was proposed as the method for electing the student body president at your school. Would you support or oppose this proposition? Write a formal letter to the editor of your school's newspaper expressing your views. Incorporate into your argument some of the features of approval voting that we considered in this section, as well as some practical considerations that might be relevant to implementing an approval voting system.

Weakening IIA: Intensity of Binary Independence

We'll conclude this section by considering a potential resolution to Arrow's Theorem that was proposed by Donald Saari, a professor of mathematics and economics at the University of California, Irvine. Saari's interpretation of Arrow's Theorem can be summarized as follows:

- We are only considering voting systems that produce transitive societal preference orders. To avoid "garbage in, garbage out" behavior, we must also require individual preferences to be transitive.
- Transitivity forces connections between pairwise comparisons in individual voter preferences. For instance, if a voter prefers A over B and B over C, then transitivity requires the voter to also prefer A over C.
- The IIA criterion requires voting systems to determine the societal preference between any pair of candidates based solely on the individual voters' preferences between those two candidates. This requirement forces voting systems to throw away the connecting information supplied by transitivity, essentially making it impossible for systems that satisfy IIA to distinguish between voters with rational, transitive preferences and voters with irrational, cyclic ones.

According to Saari, the IIA criterion essentially annihilates the assumption that individual preferences are transitive. This then causes perfectly

reasonable voting systems to be capable of producing impermissible cyclic societal preferences. Saari's solution to this problem is to weaken IIA by allowing voting systems to take into account not only the pairwise preferences of the voters in the system, but also the *intensity* with which they hold these preferences. We formalize this idea in the next two definitions.

Definition 5.33. Suppose that a voter's preference between two candidates is $A \succ B$. The **intensity** of this preference is the number of candidates listed between A and B in the voter's individual preference order.

Question 5.34.* For each of the following preference orders, what is the intensity of the voter's preference between candidates A and B?

(a) $A \succ B \succ C \succ D$

(b) $A \succ C \succ D \succ B$

(c) $D \succ A \succ C \succ B$

Definition 5.35. A voting system is said to satisfy the **intensity of binary independence** criterion (**IBI** for short) if the societal preference between any two candidates depends only on the individual voters' preferences between those two candidates *and* the intensity with which they hold these preferences.

To state Definition 5.35 in another way, with a voting system that satisfies IBI, if some or all of the voters in an election change their preference ballots, but no voter changes their preference between two specific candidates A and B or the intensity with which they hold this preference, then the societal preference between A and B will remain unchanged.

Question 5.36.* Suppose Greg, Sharon, Dean, and Carolyn are the last four competitors on the newest reality TV show, *Starvation Island*. According to the rules of the show, the Borda count is used to determine the player eliminated during each round of the contest. Unfortunately, after the ballots are cast by the four competitors, Greg experiences a moment of weakness and begins to eat them. Sharon, Dean, and Carolyn eventually manage to restrain him, but only in time to recover the following information:

- Two ballots contain the partial ordering $G \succ S$.
- One ballot contains the partial ordering $S \succ C \succ G$.
- The remaining ballot contains the full ranking $G \succ D \succ C \succ S$.

Using only this information, what can you conclude about the resulting societal preference between Greg and Sharon?

Question 5.37. Suppose that for a particular pair of candidates in an election, say A and B, you know each of the voters' preferences between these

two candidates and the intensity with which they hold these preferences. Suppose also that the voting system used in the election is the Borda count.

(a) Do you have enough information to determine the societal preference between A and B? Give a convincing argument to justify your answer. (Hint: You may find it helpful to consider again the reasoning that you used in Question 5.36.)

(b) Does the Borda count satisfy IBI? Why or why not?

Question 5.38.* Is there a voting system that satisfies universality, IBI, and unanimity, and is not a dictatorship? Would such a system contradict the strong form of Arrow's Theorem as we stated it on page 73?

Concluding Remarks

Throughout the last five chapters, we've learned a lot about voting systems and how they do or do not live up to the standards that we might reasonably expect them to. We've also seen how Arrow's Theorem tells us that certain fairness criteria are incompatible with each other—no matter what voting systems we consider. What Arrow's Theorem doesn't say, however, is that there aren't any good or reasonable voting systems to choose from. Our success in finding a voting system that behaves the way we'd want it to depends on how willing we are to compromise on certain desirable features.

Finally, it's important to note that, while we've been focusing on the mathematical properties of voting systems, practical considerations must also be taken into account. For instance, sequential pairwise voting would likely be expensive and time-consuming in elections with many candidates. And ranked systems, such as the Borda count, could also be difficult to implement in large elections. (See Question 2.32 for an example of an election with 135 candidates; can you imagine trying to rank all of them?) The fact that no voting system is perfect explains why there is so much debate about which systems should be used in various types of elections. Hopefully, our investigations up to this point have prepared you to be able to understand and evaluate the arguments put forth in these kinds of discussions, and to make your own judgments about the best ways to effectively implement democracy.

Question 5.39. Considering everything we've learned so far, which voting system do you think is the best? Give a convincing argument to justify your answer, addressing both mathematical and practical considerations. Does your answer depend on the type of election and/or number of candidates involved? If so, explain how.

Questions for Further Study

Question 5.40. Explain where we used each of the following assumptions in our proof of Arrow's Theorem:

- There are more than two candidates in the election.
- The voting system must produce transitive societal preference orders.
- The voting system must satisfy universality.
- The voting system must satisfy IIA.
- The voting system must satisfy unanimity.

Question 5.41. Which of the properties of universality, IBI, and unanimity are satisfied by Black's voting system (see Warmup 4.1), and which are not? Give a convincing argument or example to justify each of your answers.

Question 5.42. Write a short biography of Donald Saari, including his most important contributions both inside and outside of voting theory, and the voting system that he prefers.

Question 5.43. Find a copy of an article written by Donald Saari in which he uses geometric ideas to analyze voting systems, and write a summary of whatever you can understand in the article.

Question 5.44. Find some information about Steven Brams, a professor of politics at New York University, and write a summary of your findings. What voting system does he prefer? What do you think a debate between Brams and Donald Saari would be like? Assuming each made a good case for his preferred system, whose side would you take?

Question 5.45. Find out how voting is conducted for enshrinement into the National Baseball Hall of Fame, and write a detailed summary of your findings. Include in your summary a description of how nominees are selected, who votes, the voting system that is used to decide which nominees will be enshrined, the criteria failed nominees must satisfy to be held over onto the next ballot, and some actual examples for illustration.

Question 5.46.

(a) If approval voting had been used instead of plurality to determine the winner of the 2016 U.S. presidential election in New Hampshire, who do you think would have won the state? Give a convincing argument to justify your answer.

(b) If approval voting had been used instead of plurality to determine the winner of the 2000 U.S. presidential election in Florida, who do you think would have won the national election? Give a convincing argument to justify your answer.

(c) If approval voting had been used instead of plurality to determine the winner of the 1998 Minnesota gubernatorial election, who do you think would have won? Give a convincing argument to justify your answer.

Question 5.47. Propose a method of modeling approval voting (that is, a way of associating approval ballots with preference orders) so that Arrow's universality condition *is* satisfied. In your proposed model, which of Arrow's other conditions are satisfied, and which are violated?

Question 5.48. Find an editorial either online or in a popular media source that supports approval voting as a method for deciding elections with more than two candidates. Write a summary of the editorial, and compare the arguments made in it with our investigations in this chapter.

Question 5.49. Find an editorial either online or in a popular media source that supports the Borda count as a method for deciding elections with more than two candidates. Write a summary of the editorial, and compare the arguments made in it with our investigations in this chapter.

Question 5.50. Find an article or book that suggests a potential method for resolving Arrow's Theorem that is different from the three methods we investigated in this chapter. Discuss the pros and cons of the potential resolution you found, and compare it with the three methods we investigated.

Question 5.51. Some have argued that in times of emergency, short periods of dictatorship may be necessary and even desirable. Do you believe that a dictatorship can ever be beneficial to society? If so, under what circumstances? Give a convincing argument to justify your answer.

Question 5.52. Research *single-peaked* preferences, and write a summary of your findings. How can single-peaked preferences be used to resolve Arrow's Theorem, and which economist (already mentioned in a previous chapter) is known for his theorems about single-peaked preferences?

Question 5.53. Consider the following preference schedule for an election with three candidates:

	Number of Voters					
Rank	1	1	1	1	1	1
1	A	A	B	B	C	C
2	B	C	A	C	A	B
3	C	B	C	A	B	A

Suppose that approval voting is used to decide the outcome of the election, and suppose also that each voter approves of either one or two candidates. (Approving of no candidates or all three candidates is not a good strategic choice, since doing so has no impact on the outcome of the election.) Show that, depending on where each voter draws the line between approved and disapproved candidates, any societal preference order is possible.

Question 5.54. For an election with four candidates, consider a variation of the Borda count in which the points assigned are 5, 3, 1, and 0 (instead of the usual 3, 2, 1, and 0).

(a) Does this system satisfy IBI? Why or why not?

(b) In general, what conditions must be placed on the points associated with each ranking in order for the resulting system to satisfy IBI?

Question 5.55. Does approval voting satisfy the CWC? Give a convincing argument or example to justify your answer.

Answers to Starred Questions

5.4. Since B is not ranked first in the societal preference order, there must be some other candidate who is ranked above or tied with B. Likewise, since B is not ranked last, there must be some other candidate who is ranked below or tied with B.

5.6. Because every voter ranks B either first or last in their individual preference order, moving C above A with no other changes will not affect any voter's individual preference between A and B or between B and C. Thus, by IIA, the resulting societal preference between A and B and between B and C must remain unchanged.

5.8. The contradiction is that $A \succsim C$ (by Question 5.5) and $C \succ A$ (by Question 5.7). These two relations cannot both be true at the same time. Remember that we assumed that B was not ranked first or last in the societal preference order produced by V (even though every individual voter ranked B first or last). Since this assumption led to a contradiction, it must be the case that B is in fact ranked either first or last in the societal preference order produced by V. This, however, is exactly the conclusion of Lemma 5.3.

5.10. (a) If all of the voters moved B from last place to first place, then by unanimity, B would have to be ranked first in the resulting societal preference order.

(b) If only some of the voters moved B from last place to first place, then by Lemma 5.3, B would have to remain in last place or move to first place in the resulting societal preference order.

(c) Since the societal preference order would have to change—from ranking B last if every voter ranked B last to ranking B first if every voter ranked B first—as the voters moved B from last place to first place in their individual preference orders one by one, there would have to be some particular voter for which this change in the resulting societal preference order would first occur.

5.11. By IIA, none of the changes listed would affect the resulting societal preference between A and C.

5.13. Since the societal preference between A and C is the same given either S or S' (by Question 5.11), and A is preferred over C in the societal preference order produced from S' (by Question 5.12), we can conclude that A would be preferred over C in the societal preference order produced from S.

5.14. We assumed before Question 5.11 only that v_j prefers A over C. This assumption by itself allowed us to conclude (with some work) that A must also be preferred over C in the resulting societal preference order. Since we assumed nothing about the preferences of any of the other voters besides v_j, we could come to this same conclusion even if some or all of the other voters opposed v_j's preference of A over C.

5.16. The statement from part (a) is true, since v_j satisfies all three conditions. (We established this for the first two conditions, and a similar argument would work for the third condition.) We can conclude that v^* is v_j—a dictator.

5.20. To prove Lemma 5.18, we should assume that we have a voting system that satisfies monotonicity, IIA, and citizen sovereignty. We should then try to prove that this voting system also satisfies unanimity.

5.23. We can conclude that V satisfies unanimity, just as we wanted. This indeed finishes the proof of Lemma 5.18.

5.24. Plurality satisfies the modified Pareto condition. If every voter in an election preferred A over B, then B would definitely not receive any first-place votes. Hence, A could not receive fewer first-place votes than B, and so under plurality B could not be ranked higher than A in the resulting societal preference order (though A and B could be tied). Instant runoff also satisfies the modified Pareto condition, but sequential pairwise voting does not (as demonstrated by Question 4.21).

5.26. (a) Since Peter received 6 votes of approval, James 5, and John 0, the societal preference order produced by approval voting would be Peter $\succ$ James $\succ$ John.

5.27. Each of the individual preference orders in parts (a), (b), and (c) could be consistent with the two voters' approval ballots. However, the preference orders in parts (d) and (e) could not, since each involves at least one tie between a pair of candidates, one of whom was approved and the other not.

5.29. Approval voting is a voting system according to Definition 4.13, although it violates universality by placing restrictions on the kinds of preference orders that can be provided as input.

5.30. (a) Suppose that in the original election, a voter had approved of both A and B (i.e., placed both A and B in first place in their individual preference order). In the revote, the voter would have to either again approve of both A and B, or else disapprove of both A and B. In the first case, both A and B would *keep* the

approval vote they had earned from the voter in the original election. In the second case, both A and B would *lose* the approval vote they had earned from the voter in the original election. In either case, both candidates would be affected in exactly the same way, and so the overall difference in their approval votes would not change. The same would be true if the voter had originally disapproved of both A and B, or if the voter had originally approved of one and disapproved of the other.

(b) Part (a) allows you to conclude that approval voting satisfies IIA.

5.31. Approval voting is anonymous, neutral, and monotone, *and* it satisfies the Pareto condition.

5.34. (a) Since no candidates are listed between A and B, the intensity of the $A \succ B$ preference is 0.

(b) Since two candidates (C and D) are listed between A and B, the intensity of the $A \succ B$ preference is 2.

5.36. On each of the two ballots containing the partial ordering $G \succ S$, Greg would receive 1 more point than Sharon. On the ballot containing the partial ordering $S \succ C \succ G$, Sharon would receive 2 more points than Greg. On the remaining ballot, Greg would receive 3 more points than Sharon. Therefore, the total point differential between Greg and Sharon would be 3 points in favor of Greg, yielding the societal preference $G \succ S$.

5.38. The Borda count satisfies universality, IBI, and unanimity. Clearly, the Borda count is not a dictatorship. This does not contradict the strong form of Arrow's Theorem though, since IBI is a weaker criterion than IIA.

Chapter 6

Gaming the System

Focus Questions

In this chapter, we'll explore the following questions:

- What is strategic voting?
- What are some common types of strategic voting for various voting systems?
- What does the Gibbard-Satterthwaite Theorem say about manipulation of voting systems?
- How is the Gibbard-Satterthwaite Theorem related to Arrow's Theorem? How can it be proved?

Warmup 6.1. The emergence of a competitive third party has complicated the previously straightforward mayoral elections in Stickeyville. In the upcoming election, three candidates are running: Deion Davis (a Democrat), Rachel Ramirez (a Republican), and Gabriella Gardner (of the Green Party). The preferences of the residents of Stickeyville are as shown in Table 6.1.

	Number of Voters			
Rank	25	15	15	45
1	G	D	D	R
2	D	G	R	D
3	R	R	G	G

TABLE 6.1. Preference schedule for Stickeyville

(a) If the residents of Stickeyville used instant runoff to decide the outcome of the election, who would win, and what would the resulting societal preference order be?

(b) Suppose that five of the $D \succ G \succ R$ voters decide that they actually like Gardner best and change their preferences accordingly, to $G \succ D \succ R$. Would this change result in a better or worse outcome for these voters?

(c) Suppose that six of Ramirez's 45 voters change their preferences to $G \succ R \succ D$. How would this change impact the outcome of the election?

Throughout most of our investigations so far, we have assumed that voters will vote *sincerely*. In other words, we have assumed that each voter will submit a ballot that represents their true preferences. However, as we saw in Warmup 6.1, there are situations in which it could benefit a voter to misrepresent their preferences. In part (b), the voters who switched to Gardner would have been better off sticking with Davis, even though he was no longer their first choice. In part (c), we observed something even more devious, where just a small number of Ramirez's voters were able to exploit the sequential nature of the instant runoff system to secure a victory for their candidate. They accomplished this by adopting the unexpected strategy of ranking Gardner—their *last* choice—first. It was a gamble that paid off, as the extra support for Gardner caused Davis to be eliminated in the first round, leading to a solid victory for Ramirez in the second round.

What these examples demonstrate is that instant runoff is *manipulable*. And, as it turns out, most voting systems are. In this chapter, we'll look at some common strategies for manipulating the voting systems we've studied in previous chapters. We'll also explore an important result that tells us exactly which voting systems are immune to manipulation.

Strategic Voting

In a ranked voting system, it can be hard enough for voters to figure out exactly what their preferences are, particularly if there are a lot of candidates. When a voting system is manipulable, however, the choices become even more difficult. It's no longer a matter of just deciding which candidates are better or worse; voters must also decide whether to vote in accordance with their actual preferences, or to cast a strategic ballot in the hopes of securing a better election outcome.

There are a number of different tactics that strategic voters might adopt, and we've seen examples of several of them already. Some of the most common strategies include:

- **Compromising**, in which a voter casts a ballot for a candidate other than their top choice, usually because they perceive the compromise candidate to be more electable.
- **Pushover** or **mischief voting**, in which a voter ranks a candidate higher than they actually prefer, but not with the intent of electing

that candidate. Instead, the aim is to indirectly benefit the voter's most preferred candidate by harming their opponents.

- **Burying**, in which a voter ranks a strong opponent lower than they actually prefer in order to have a better chance of defeating that opponent.
- **Bullet voting**, in which a voter has the option to vote for multiple candidates (as in approval voting), but votes for only one in order to deny support to opposing candidates.

Question 6.2.* Which of the strategies described above were illustrated in parts (b) and (c) of Warmup 6.1?

Question 6.3. Consider the main voting systems we have studied so far: majority rule, plurality, the Borda count, sequential pairwise voting, instant runoff, and approval voting.

(a) Which of these systems do you think are most susceptible to compromising, and why?

(b) Which systems do you think are most susceptible to pushover voting, and why?

(c) Which systems do you think are most susceptible to burying, and why?

Question 6.4.* Suppose approval voting had been used for the election from Warmup 6.1. Give an example to illustrate how bullet voting could have been a smart strategic choice for some of the voters.

The Gibbard-Satterthwaite Theorem

Researchers have spent quite a bit of time studying manipulability of voting systems, in part because avoiding the potential for manipulation seems a perfectly natural and desirable goal. We would like to think that election outcomes can be determined solely by voter preferences and possibly the voting system used to aggregate the ballots. But if voters can sway the results of an election by voting strategically, some will do so. On the other hand, voters who are not as informed about the strategic choices available to them may cast ballots that are not effective—or they might disengage from the political process altogether. What would be ideal is if we could take the effects of strategic voting out of the equation by choosing a system in which all voters would have an incentive to simply be honest and vote according to their actual preferences.

The good news is that some voting systems are in fact immune to manipulation, including some that we studied back in Chapter 1.

Question 6.5. Explain why in each of the following systems, voters have no incentive to vote insincerely.

(a) Majority rule with exactly two candidates

(b) Dictatorship with any number of candidates

(c) Imposed rule with any number of candidates

The voting systems in Question 6.5 are probably not the ones you were hoping for. After all, we already know that elections with only two candidates are pretty straightforward, and there are not many proponents of democracy who would seriously advocate for a dictatorship or imposed rule. So what other options are there?

The short answer to this question is... there aren't any. We have philosopher Allan Gibbard and economist Mark Satterthwaite to thank for this not-so-happy result. They proved, independently, in 1973 (Gibbard) and 1975 (Satterthwaite) that a dictatorship is the only non-manipulable voting system for three or more candidates that: (1) works with any set of transitive preferences; (2) always produces a unique winner; and (3) can produce any candidate as the winner.

In order to formally state the Gibbard-Satterthwaite Theorem, we'll need to define a few terms more precisely. First off, the theorem doesn't deal with voting systems exactly as we've defined them in past chapters. Instead, it is concerned only with selecting a winner—not a societal preference order—for each set of voter preferences. To make this distinction clear, we'll adopt the following definition:

Definition 6.6. A **choice function** is a function that receives as input any collection of transitive preference ballots and produces as output a unique winning candidate.

We will be interested in finding choice functions that cannot be manipulated. But what exactly does it mean for a choice function to be manipulable? The next definition provides an answer.

Definition 6.7. A choice function is **manipulable** if there is a situation in which some voter can singlehandedly ensure a more desirable outcome (from their perspective) by submitting a ballot that is different from their true preferences.

When we talk about manipulation, we are always looking at voting situations in which a single voter changes their ballot, while the remaining ballots stay fixed. The idea is that manipulation does not require cooperation—or at least there are some situations where only one voter changing their ballot is enough to change the outcome of the election.

We also need to define what a dictator is with respect to a choice function.

Definition 6.8. For a given choice function, a **dictator** is a voter who can force any candidate to win by ranking that candidate first (without ties) on

their ballot. A choice function is said to be **dictatorial** if some voter is a dictator with respect to it.

Finally, we need to say precisely what it means for a choice function to be able to produce any candidate as the winner. We'll capture this idea with a definition similar to that of citizen sovereignty in Arrow's Theorem.

Definition 6.9. A choice function is **non-imposed** if for every candidate there is a set of ballots for which the choice function will produce that candidate as the winner.

With these definitions in mind, we can now state the Gibbard-Satterthwaite Theorem.

Gibbard-Satterthwaite Theorem. *There does not exist a choice function for an election with more than two candidates that is non-imposed, non-manipulable, and non-dictatorial.*

Proving the Gibbard-Satterthwaite Theorem

If the Gibbard-Satterthwaite Theorem reminds you of Arrow's Theorem, you're wise to observe the similarities between the two. Both are impossibility theorems, both deal with elections having more than two candidates, and both involve non-imposition and non-dictatorship conditions. In light of these similarities, it may not surprise you to learn that the Gibbard-Satterthwaite Theorem is actually a corollary of Arrow's Theorem, meaning that we can use Arrow's Theorem to prove it. For the rest of the chapter, that's what we'll do. The proof is not easy, but—like the proof of Arrow's Theorem—we'll be able to understand it by breaking it down into smaller steps.[1]

Our main strategy will be to show that any non-imposed, non-manipulable choice function for an election with three or more candidates must in fact be dictatorial. We'll do so by showing that any such choice function can be extended to a voting system that satisfies universality, unanimity, and IIA. By the strong form of Arrow's Theorem, we will then be able to conclude that the voting system we constructed must be a dictatorship. Finally, we'll use this conclusion to show that the original choice function must also be dictatorial.

Monotonicity and Non-Manipulability

Before we go any further, there is one more useful property that we need to consider. It is a version of the monotonicity property we've studied in previous chapters, but this time with regard to choice functions.

[11] Our approach is based on that of Vazirani et al. [**56**] in their book *Algorithmic Game Theory*.

Definition 6.10. A choice function is said to be **monotone** if the only way a single voter can cause the winner to change—say from A to B—is for that voter to change their ranking of A and B from $A \succ B$ to $B \succ A$.

Notice that this definition of monotonicity is a little stronger than the one we've used in the past. In addition to saying that a change favorable to A cannot cause the winner to change from A to B, it says that in order for a voter to singlehandedly force the winner to change from A to B, that voter must make a specific change favorable to B—namely, they must change their ranking of A and B from $A \succ B$ to $B \succ A$. We will use this stronger definition in order to more easily see a relationship between monotonicity and manipulability.

To that end, note that if a choice function is not monotone, this means that there are two sets of ballots for which: (1) the choice function produces different winners—say A for the first set of ballots and B for the second set; (2) the two sets of ballots are identical except for one special voter; and (3) this special voter ranks A and B the same in both sets of ballots (either $A \succ B$ or $B \succ A$).

Question 6.11.* Explain why a choice function that is not monotone must be manipulable.

Worded slightly differently (but equivalently), Question 6.11 establishes the following lemma:

Lemma 6.12. *Every non-manipulable choice function is monotone.*

Choice Functions and Voting Systems

Now let's continue with our proof of the Gibbard-Satterthwaite Theorem. We'll start with any choice function for three or more candidates that is non-imposed and non-manipulable. Let's call that function f (a letter mathematicians often like to use to denote functions). We're now going to try to use f to define a voting system.

Remember that the difference between a choice function and a voting system is that a voting system must output a transitive societal preference order, whereas a choice function just gives us a winner. In other words, we need to *extend* f so that it gives us not just a winner, but a ranking of the candidates. Here's how we'll do it:

- To decide the societal ranking between any pair of candidates, say A and B, we'll first move A and B to the top of every voter's ballot. We won't change any other rankings, and we won't change the voters' relative rankings *between* A and B; if a voter prefers one to the other, it will stay that way, and if a voter is indifferent between A and B, they will remain tied. What will change is that A and B will both be ranked strictly higher than every other candidate.

- Once we've moved A and B to the top of every voter's ballot, we'll ask f who it will choose as the winner. If f chooses A, we'll say society prefers A to B. If f chooses B, we'll say society prefers B to A.
- After we've considered every possible pair of candidates, we'll combine these pairwise rankings to get a societal preference order.
- We'll let F denote the voting system that sends each set of ballots to the societal preference order constructed as described above.

To illustrate, let's consider an example.

Question 6.13.* Assume that the voters in an election are arranged in some order, say v_1, v_2, ..., v_n, and let f be the choice function that works as follows:

- If v_1 has a unique top-ranked candidate, then that candidate is declared the winner. Otherwise, the candidates who are not among v_1's top choices are eliminated.
- If v_2 has a unique top-ranked candidate among those who remain, then that candidate is declared the winner. Otherwise, the remaining candidates who are not among v_2's top choices are eliminated.
- The process continues in this same manner, with each successive voter eliminating candidates until only one remains. If multiple candidates still remain after the last voter, then the tie is broken by alphabetical order, with the candidate whose name comes first in the alphabet being declared the winner.

Now suppose that there are three voters in the election, with the following preferences:

- $v_1 : A \approx B \succ C \approx D$
- $v_2 : B \succ C \approx D \succ A$
- $v_3 : D \succ C \succ A \succ B$

Given these ballots, what societal preference order would F produce?

The Nitty-Gritty Details

There are a couple of details we need to check in order to make sure that F, as we've defined it above, always produces a transitive societal preference order.

First, in constructing F, we implicitly assumed that whenever two candidates are moved to the top of every voter's ballot, f will choose one of those two candidates as the winner. But what if this didn't happen? What if, for example, we moved A and B to the top of every ballot, but f chose some other candidate C as the winner? In this case, we wouldn't be able to write down a ranking of A and B. Fortunately, however, the next lemma rules out this possibility.

Lemma 6.14. *If f is a non-imposed, non-manipulable choice function, and if there is a set S of candidates that are at the top of every voter's preference order (meaning that every voter prefers every candidate in S to every candidate not in S), then f must choose a candidate in S as the winner.*

In the next question, we'll develop an argument to show why Lemma 6.14 is true. It may take a little while to fully understand this argument, but it's important to put in the effort, since we'll use similar arguments throughout the rest of the proof of the Gibbard-Satterthwaite Theorem. To make matters somewhat simpler, we'll look at the special case of the lemma in which the set S has exactly two candidates—say A and B—in it.

Question 6.15.* Let b_1, b_2, b_3, ..., b_n be a set of ballots where candidates A and B are at the top of every ballot.

(a) Explain why there must be some other set of ballots, say b_1^*, b_2^*, b_3^*, ..., b_n^*, such that given those ballots, f would choose A as the winner.

(b) What would happen if we gave f the ballots $b_1, b_2^*, b_3^*, \ldots, b_n^*$? Could f choose a candidate other than A or B as the winner? Why or why not? (Hint: Only one voter's ballot is different from those in part (a). What would monotonicity require in this situation for the winner to change from A to some other candidate C?)

(c) What would happen if we gave f the ballots $b_1, b_2, b_3^*, \ldots, b_n^*$? What possible winners could f choose?

(d) If we continue this process, changing one ballot at a time, we will eventually get back to our original set of ballots: $b_1, b_2, b_3, \ldots, b_n$. Explain why this observation establishes that, given these ballots, f must choose A or B as the winner.

Lemma 6.14 tells us that our method for extending a choice function f to a voting system F will, at the very least, produce a ranking of every pair of candidates. Now all we need to do is combine those pairwise rankings to get a societal preference order. But what if the pairwise rankings don't fit together nicely? What if we end up with $A \succ B$, $B \succ C$, and $C \succ A$—a (gasp!) cycle? Apart from having some serious déjà vu from our investigations of Condorcet's paradox (see Question 3.4), this would really derail our efforts. After all, voting systems are supposed to produce *transitive* societal preference orders, and cycles are exactly the opposite of that. The good news is that we're lucky again: We'll never actually end up with a cycle when we follow the method we've been using. The next question shows why.

Question 6.16. Suppose that for some set of ballots, say $b_1, b_2, \ldots, b_n$, the rankings produced by F combine to give the cycle $A \succ B \succ C \succ A$. If we move A, B, and C to the top of each of these ballots, we'll get a new set of

ballots, which we'll denote by

$$b_1^{ABC}, b_2^{ABC}, \ldots, b_n^{ABC}.$$

Lemma 6.14 tells us that, given these new ballots, f must choose A, B, or C as the winner. For the sake of our argument, we'll suppose f chooses A. (We could make a similar argument if f chose B or C instead.)

(a) Suppose that instead of moving A, B, and C to the top of the first ballot, we had just moved A and C to the top. We'll call that ballot b_1^{AC}. What winner would f choose if we gave it the ballots $b_1^{AC}, b_2^{ABC}, \ldots, b_n^{ABC}$? (Hint: How would b_1^{AC} and b_1^{ABC} have to be different in order for f to no longer choose A? Is this possible?)

(b) Suppose we continue changing ballots, one by one (as in Question 6.15), until we obtain $b_1^{AC}, b_2^{AC}, \ldots, b_n^{AC}$. What winner will f choose for this set of ballots?

(c) What does your answer to part (b) tell you about how F (the voting system) will rank A and C? Why is this a problem given our assumption that $A \succ B \succ C \succ A$?

The type of argument we used in Question 6.16 is called a *proof by contradiction*. If we assume that F produces a cycle, $A \succ B \succ C \succ A$, then we can show that it must also be the case that $A \succ C$. But we can't have both $C \succ A$ and $A \succ C$ at the same time—that would be a contradiction! And since we can never derive a contradiction from a true statement (as long as our logic is correct), this means that our original assumption that F can produce a cycle must be false.

Let's pause for a moment to catch our breath and take stock of what we've accomplished and where we're headed next. We've shown that we can extend any non-imposed, non-manipulable choice function f to a voting system F, and we've ensured that F always produces a transitive societal preference order. Combined with the fact that f, as a choice function, accepts *any* set of transitive preference ballots, we have actually shown that our voting system F satisfies Arrow's universality condition. Our goal is to show that F also satisfies unanimity and IIA, and is therefore—by Arrow's Theorem—a dictatorship.

Question 6.17. To prove that F satisfies unanimity, we'll assume that every voter ranks A above B on their ballot.

(a) Suppose we move A and B to the top of every voter's ballot. Explain why A must then be the unique top-ranked candidate on every ballot.

(b) In light of the observation from part (a), what does Lemma 6.14 allow us to conclude about the candidate chosen by f?

(c) Use your answer to part (b) to explain why F produces the societal ranking $A \succ B$ and therefore satisfies unanimity.

Question 6.18. Now let's show that F satisfies IIA. Consider two sets of ballots: $b_1, b_2, \ldots, b_n$ and $b_1^*, b_2^*, \ldots, b_n^*$. Suppose also that both sets of ballots agree on their rankings of two candidates, say A and B. That is, suppose that the ranking of A and B on any ballot b_i is the same as on b_i^*.

(a) Now move A and B to the top of each ballot to obtain

$$b_1^{AB}, b_2^{AB}, \ldots, b_n^{AB}$$

and

$$(b_1^*)^{AB}, (b_2^*)^{AB}, \ldots, (b_n^*)^{AB}.$$

Explain why these new sets of ballots must still agree on their rankings of A and B. In other words, explain why the ranking of A and B on b_i^{AB} is always the same as on $(b_i^*)^{AB}$.

(b) Explain why, given either set of ballots—$b_1^{AB}, b_2^{AB}, \ldots, b_n^{AB}$ or $(b_1^*)^{AB}, (b_2^*)^{AB}, \ldots, (b_n^*)^{AB}$—$f$ must choose either A or B as the winner.

(c) We want to argue that f chooses the same candidate when given $b_1^{AB}, b_2^{AB}, \ldots, b_n^{AB}$ as when given $(b_1^*)^{AB}, (b_2^*)^{AB}, \ldots, (b_n^*)^{AB}$. To do so, we will sequentially (one voter at a time) change the first set of ballots into the second, as we have done in previous questions. To start, explain why f will choose the same winner when given $(b_1^*)^{AB}, b_2^{AB}, \ldots, b_n^{AB}$ as when given $b_1^{AB}, b_2^{AB}, \ldots, b_n^{AB}$. (Hint: Only one ballot is potentially different between the two sets, and the difference is limited by your answer to part (a).)

(d) Now continue changing ballots, one at a time, from b_i^{AB} to $(b_i^*)^{AB}$. Explain why each such change has no impact on the winner chosen by f.

(e) Use part (d) to argue that f will choose the same winner given either $b_1^{AB}, b_2^{AB}, \ldots, b_n^{AB}$ or $(b_1^*)^{AB}, (b_2^*)^{AB}, \ldots, (b_n^*)^{AB}$. Explain why this implies that F ranks A and B the same way given either set of ballots, and thus F satisfies IIA.

We have now shown that F is a voting system that satisfies universality, unanimity, and IIA. But we know from Arrow's Theorem that any such voting system must be a dictatorship. This means that there is some voter, say v, whose ballot always agrees with the societal preference order given by F. Does this make v a dictator for f as well? Let's find out.

Question 6.19.* Let v be the dictator for the voting system F, and suppose that A is v's unique top-ranked candidate. We want to see if f will always choose A as the winner. To do so, suppose that there is some set of ballots in which v ranks A first but f chooses another candidate, say B, instead.

(a) What kind of proof are we setting up by making this assumption?

(b) Suppose we move A and B to the top of each ballot. Explain why f will choose A in this situation. (Hint: Remember that v is a

dictator for F and ranks A above B. Also remember how F is defined.)

(c) Now, starting with the ballots from part (b), change one ballot at a time to put A and B back where they were before we moved them to the top. Explain why none of these changes can cause f to choose B rather than A.

(d) Explain why part (c) leads to a contradiction.

Question 6.19 finishes the proof of the Gibbard-Satterthwaite Theorem. Under the assumption that our choice function f is non-imposed and non-manipulable, we were able to find a dictator for f. This means that f cannot possibly be non-imposed, non-manipulable, *and* non-dictatorial.

Concluding Remarks

Like Arrow's Theorem, the Gibbard-Satterthwaite Theorem can be viewed as a disappointing result for the practice of democracy. In some sense, it confirms what we've seen in a variety of other settings—namely, that voting systems can sometimes produce unexpected outcomes. Naturally, there are times when the quirks of a particular system can be exploited by voters to give an advantage to their preferred candidates.

Also like Arrow's Theorem, there are resolutions to the Gibbard-Satterthwaite Theorem that make its conclusions seem somewhat less dire. Consider, for example, approval voting. If voters are really only interested in getting one of their approved candidates selected—and they don't like certain approved candidates better than others—then the only reasonable strategy is to vote for all of their approved candidates. The Gibbard-Satterthwaite Theorem doesn't apply in this setting, since we are restricting voters to having *dichotomous* preferences, as we did in Chapter 5. In other words, we are viewing approval voting as a sort of choice function that only accepts certain types of ballots—those that rank one group of approved candidates above one group of disapproved candidates, with no preferences expressed among the candidates within each group. Since our definition of a choice function doesn't allow this kind of restriction (analogous to the fact that our model of approval voting from Chapter 5 violates universality), we avoid Gibbard-Satterthwaite's unpleasant conclusions.

If we assume, however, that voters have additional preferences underlying their approval ballots, then a version of the Gibbard-Satterthwaite Theorem applies,[2] and voters must make the strategic decision of where to draw the line between approved and disapproved candidates. There might be multiple approval ballots that could all be viewed as sincere representations of the voter's preferences. For example, a voter with preferences

[22]Gibbard's version of the theorem allows for a more complex model of ballots and strategies that accommodates mixing preference rankings with approval voting strategies.

$A \succ B \succ C$ could reasonably approve only of A, or of A and B, or of A, B, and C (although the latter would be a wasted ballot, since it would have no impact on the outcome of the election). Voting for only A (bullet voting) increases the chances of electing the voter's most preferred candidate but may also lead to a less desirable outcome if voters who prefer $C \succ B \succ A$ adopt a similar strategy. In this case, a compromise candidate such as B may receive very few approval votes but actually be viewed as acceptable by the entire electorate, had they voted sincerely.

Questions for Further Study

Question 6.20. Write a short biography of Allan Gibbard, including some of his contributions apart from the Gibbard-Satterthwaite Theorem.

Question 6.21. Write a short biography of Mark Satterthwaite, including some of his contributions apart from the Gibbard-Satterthwaite Theorem.

Question 6.22. Give an argument to explain why approval voting with dichotomous preferences is immune to manipulation.

Question 6.23. Give an example to show how sequential pairwise voting is subject to pushover voting. In your example, does it make a difference if you allow voters to change their ballots between rounds of voting?

Question 6.24. Explain why burying is not an effective strategy in elections decided by instant runoff. Give specific examples to illustrate your argument.

Question 6.25. Explain why pushover voting is not an effective strategy in elections decided by the Borda count. Give specific examples to illustrate your argument.

Question 6.26. Suppose that in an election decided by approval voting, you prefer candidate A over candidate B. Is there ever a situation in which it makes sense to approve of B and disapprove of A? Give a convincing argument or example to justify your answer.

Question 6.27. Research Duverger's Law, and write a brief summary of your findings. Be sure to explain how Duverger's Law is related to manipulation and strategic voting.

Question 6.28. Research the participation criterion, and write a brief summary of your findings. Be sure to explain how the participation criterion is related to manipulation and strategic voting.

Question 6.29. Where in the proof of the Gibbard-Satterthwaite Theorem did we use the assumption that there are three or more candidates in the election? Why is this assumption essential to the theorem?

Question 6.30. Find an article or online source arguing that instant runoff is less manipulable than approval voting. Then find a source that argues the opposite. Summarize the key points made by each, and explain which argument you find more persuasive.

Answers to Starred Questions

6.2. Part (b) illustrated how *compromising*, by voting $D \succ G \succ R$, would have resulted in a better outcome for the Gardner converts than voting according to their true preferences, $G \succ D \succ R$. Part (c) gave an example of *pushover voting*, since the Ramirez voters who voted for Gardner did so not to help her win, but rather to harm Ramirez's main opponent, Davis. The strategy was effective, since Davis ended up being eliminated in the first round, ensuring an easy victory for Ramirez in the second round.

6.4. Suppose every voter approves of their top two choices. Then Davis will win the election with 100 approval votes, followed by Ramirez with 60 and Gardner with 40. However, if the Ramirez voters employ a bullet voting strategy and only approve of Ramirez, then Ramirez will win with 60 approval votes, followed by Davis with 55 and Gardner with 40.

6.11. Continuing the argument from the preceding paragraph, suppose that the special voter prefers B over A in both ballots. Since the first set of ballots produces A as the winner, whereas the second set produces B as the winner, this voter could singlehandedly ensure a better election outcome (from their perspective) by voting the second ballot instead of the first. Likewise, if the voter prefers A over B in both ballots, they can ensure a better election outcome by voting the first ballot instead of the second.

6.13. We'll need to consider each pair of candidates. For A and B, moving these candidates to the top of each voter's ballot yields the following preferences:

- $v_1 : A \approx B \succ C \approx D$
- $v_2 : B \succ A \succ C \approx D$
- $v_3 : A \succ B \succ D \succ C$

Now we need to see which candidate f will choose with these ballots. Since A and B are the top-ranked candidates on v_1's ballot, f eliminates C and D before moving on to v_2. Since v_2 prefers B to A, f chooses B as the winner. Therefore, F produces a societal ranking of $B \succ A$.

Looking at another pair, say C and D, we get the following ballots:

- $v_1 : C \approx D \succ A \approx B$
- $v_2 : C \approx D \succ B \succ A$
- $v_3 : D \succ C \succ A \succ B$

For these ballots, f eliminates A and B first, since they are not among v_1's top-ranked candidates. Moving on to v_2, f is unable to choose a winner since v_2 is indifferent between C and D. Finally, v_3 breaks the tie, resulting in a societal ranking of $D \succ C$.

Repeating this process for every possible pair of candidates yields the rankings $B \succ A$, $A \succ C$, $A \succ D$, $B \succ C$, $B \succ D$, and $D \succ C$. Combining these rankings yields the societal preference order $B \succ A \succ D \succ C$.

6.15. (a) Since f is non-imposed, there must be some collection of ballots for which f chooses A as the winner.

(b) The only difference between the ballots in part (a) and those in part (b) is that the first ballot is b_1 instead of b_1^*. Because f is monotone, for the winner to change from A to C, the first voter would have had to change their ranking of A and C from $A \succ C$ to $C \succ A$. But it can't be the case that $C \succ A$ in b_1, since A and B are top-ranked in b_1. Therefore f can't choose C as the winner.

(c) By the same argument as in part (b), since we only changed one ballot—from b_2^* to b_2—and A and B are both top-ranked in b_2, f couldn't choose any winner other than A or B.

(d) Each time we change a ballot from b_i^* to b_i, f still must choose either A or B as the winner. By the time we have completed the sequence of changes, we'll have the ballots $b_1, b_2, \ldots, b_n$, and f will still have to choose either A or B as the winner.

6.19. (a) We are setting up a proof by contradiction.

(b) Moving A and B to the top of each ballot will not change the fact that v ranks A above B. Since the societal preference order produced by F must agree with v, it must yield a ranking of $A \succ B$. But because of the way F is defined, this means that f must choose A.

(c) Moving A and B back to where they began on a ballot does not change the relative ranking of A and B on that ballot—only how A and B compare to the remaining candidates. By monotonicity, it is therefore impossible for such a change to cause f to choose B, rather than A, as the winner.

(d) If we move A and B back, one ballot at a time, to their original positions, we will eventually end up with our original set of ballots. But none of the changes we make along the way will cause f to choose B as the winner. This is a contradiction, since we assumed that f would choose B as the winner given the original set of ballots.

Chapter 7

One Person, One Vote?

Focus Questions

In this chapter, we'll explore the following questions:

- What is a weighted voting system? What are some common examples of weighted systems?
- How are yes/no voting systems similar to the voting systems we investigated in previous chapters? How are they different?
- What does it mean for a voter in a yes/no voting system to be a dictator or to have veto power? Are these properties equivalent?
- How can the properties of swap and trade robustness be used to characterize yes/no voting systems?

Warmup 7.1. After a particularly unsuccessful advertising campaign, the three shareholders of Captain Ahab's Fish & Chips have called an emergency meeting to decide the fate of their vice president of marketing, Deanne Boomhauer. The number of shares of stock held by each shareholder is shown in Table 7.1.

Shareholder	Shares
Doug	101
Nicholas	97
Elisabeth	2

TABLE 7.1. The shareholders of Captain Ahab's Fish & Chips

At the start of the meeting, Doug makes a proposal to fire Boomhauer and search for a replacement. A vote is called, and each shareholder is to vote either *yes* or *no* on Doug's proposal.

(a) Would majority rule be an appropriate method to use to determine the outcome of the voting? Why or why not?

(b) What method do you think should be used to determine the outcome of the voting? What do you think would happen to Boomhauer if this method were used?

Back in Chapter 1, we identified anonymity (treating all voters equally) as one of the essential properties that voting systems should satisfy. The idea of "one person, one vote" seems to be inextricably linked to our notion of democracy, and rightfully so. But, as we saw in Warmup 7.1, there are situations in which it is not appropriate for the preferences of all of the voters in an election to carry the same weight. In the corporate world, for example, shareholders who own more stock in a company may rightfully deserve a greater say in the company's operating decisions than those who own less stock. In this case, the inequity seems entirely reasonable—after all, it wouldn't make sense for someone who owns only one or two shares of stock to have as much clout as someone who owns half the company.

In answering part (b) of Warmup 7.1, you may have suggested that each shareholder's vote should be weighted in some way that is proportional to the number of shares of stock they own. Voting systems that operate according to this principle are naturally called *weighted* voting systems. They typically arise in situations in which a yes/no or pass/fail decision is required on some motion or proposal. It's important to note the contrast between this type of election and the elections we've considered in previous chapters. There, we were interested in forming some kind of ranking of candidates for an elected office (or at least in choosing a winning candidate). Here, our primary interest will be in deciding the answer to some yes/no question—for example, "Should Boomhauer be fired and a replacement sought?" That's not to say that weighted voting systems don't arise in other contexts. In fact, the system used to elect the president of the United States—the Electoral College—is a type of weighted voting system, which we'll consider in detail in Chapter 9. For now, however, our focus will be on weighted voting systems within the context of yes/no decisions.

Weighted Voting Systems

Definition 7.2. A **yes/no voting system** is a system used to make a decision on a yes/no question, or *motion*. A **weighted voting system** is a yes/no system characterized by the following:

- A collection of voters.
- A collection of **weights**. In particular, we associate with each voter a positive number called the voter's *weight*, which is understood to be the number of votes controlled by the voter.
- A **quota**. The quota is a positive number q such that a motion will *pass* if the sum of the weights of the voters who vote "yes" on the motion equals or exceeds q (and the motion will *fail* otherwise).

There's a lot going on in Definition 7.2, so let's look back at Warmup 7.1 to put it into a natural context.

Question 7.3.* Suppose that a weighted voting system is to be used to decide the outcome of Doug's motion from Warmup 7.1 (to fire Boomhauer and search for a replacement).

(a) Who would the voters in this system be?

(b) What weight should be assigned to each voter?

(c) What would a reasonable choice for the quota be, and why?

Question 7.4.* For parts (a)–(c) below, use the voters and weights that you specified in the first two parts of Question 7.3.

(a) If the quota was 101, which combinations of voters could cause Doug's motion to pass by voting in favor of it? With this quota, what distinguishes Doug from the other voters?

(b) Repeat part (a), but with a quota of 103.

(c) Repeat part (a), but with a quota of 105.

Question 7.4 illustrates several important features of weighted voting systems—and, by extension, yes/no systems in general. First, notice that in a yes/no voting system, whether a motion passes or fails may depend not necessarily on *how many* voters vote in favor of it, but rather on *which* voters vote in favor of it. This fact motivates the following definition.

Definition 7.5. The following definitions apply to yes/no voting systems.

- A **coalition** is a collection of any number of voters, ranging from no voters (the *empty* coalition) to all of the voters in the system.
- A **winning coalition** is a coalition that can force a motion to pass by voting in favor of it. In other words, if every member of a winning coalition votes in favor of a motion, it will pass—even if every voter outside the coalition votes against it.
- A **losing coalition** is a coalition that cannot force a motion to pass by voting in favor of it. In other words, even if every member of a losing coalition were to vote in favor of a motion, the motion could still fail.
- A **minimal winning coalition** is a winning coalition that would become a losing coalition if any individual voter were removed from it.
- In a weighted voting system, the **weight** of a coalition is the sum of the weights of the voters in the coalition.

It's worth noting that if we were only ever going to be interested in weighted voting systems, we could have defined a winning coalition as *a coalition whose weight is greater than or equal to the quota*, and a losing

coalition as *a coalition whose weight is less than the quota*. These phrasings are obviously more concise, but also somewhat limiting; our more general definitions will allow us to talk about winning and losing coalitions even when we are considering yes/no voting systems that are not necessarily weighted.

Question 7.6.* For each part of Question 7.4, list all of the winning coalitions, minimal winning coalitions, and losing coalitions.

Question 7.7. Suppose the quota in part (c) of Question 7.4 was 150 instead of 105. Would this change affect any of the winning or minimal winning coalitions for the system? Why or why not?

Question 7.7 shows that, in some cases, two different weighted voting systems can have the exact same winning coalitions. When this occurs, it is natural to say that the two systems are essentially the same, or, to use a favorite word of mathematicians, *isomorphic*.[1]

Definition 7.8. Two weighted voting systems are said to be **isomorphic** if they have the exact same winning coalitions.

In a weighted voting system with n voters, it is common to use the notation $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$ to represent the voters and their weights, ordered from largest weight to smallest. We can then use the shorthand $[q : w_1, w_2, \ldots, w_n]$ to describe a weighted voting system with weights $w_1, w_2, \ldots, w_n$ and quota q. So, for example, to describe the system from part (b) of Question 7.4, we would write $[103 : 101, 97, 2]$. With this notation in hand, the next question should help you to better understand what it means for two weighted voting systems to be isomorphic.

Question 7.9.* For each of the following weighted voting systems, list all of the winning coalitions. Then decide which of the systems are isomorphic.

(a) $[4 : 2, 2, 1]$

(b) $[4 : 3, 2, 1]$

(c) $[5 : 3, 2, 1]$

(d) $[5 : 3, 2, 2]$

(e) $[5 : 3, 3, 2]$

[1] As a word of caution, as you become more educated in the language of mathematics, you may be tempted to use words such as this in everyday conversation. For example, when discussing your favorite sorority sisters, who happen to be identical twins: "What, you were talking about Julie? I thought you were talking about Michelle! Oh well, they're isomorphic anyways! Ha, ha, ha!" Be forewarned that, while some of your more mathematically inclined friends may get a good laugh out of this, others may not appreciate your newfound sense of humor.

Dictators, Dummies, and Veto Power

Now that we've defined the various types of coalitions and explored what it means for two weighted voting systems to be isomorphic, let's revisit Question 7.4 one more time. In part (a) of that question, you probably noticed that Doug was a dictator in the sense that the outcome of the voting would always be identical to however he voted. Put another way, Doug was a dictator because he was present in every winning coalition and absent from every losing coalition.

In part (b), however, Doug had slightly less power. While he could still keep a motion from passing by voting against it, he did not have enough weight to singlehandedly force a motion to pass by voting in favor of it. In this situation, we would say that Doug had *veto power*. Put another way, Doug had veto power in part (b) because he was present in every winning coalition. (Notice though that Doug was *not* a dictator in part (b), since he was also present in the losing coalition {Doug}.)

In part (c), Doug's power was even less; he and Nicholas, despite having different weights, were actually equally powerful. Furthermore, poor Elisabeth was completely powerless, unable to affect the outcome in any way, no matter how she voted. In this situation, the somewhat unfortunate convention is to call Elisabeth a *dummy*. Using the language of coalitions, Elisabeth was a dummy in part (c) because she could be removed from any winning coalition to which she belonged, and the coalition would still be winning. Put another way, Elisabeth was a dummy because she was not present in any minimal winning coalitions.

Definition 7.10. In any yes/no voting system:

- A voter who is present in every winning coalition and absent from every losing coalition is called a **dictator**.
- A voter who is present in every winning coalition is said to have **veto power**.
- A voter who is not present in any minimal winning coalitions is called a **dummy**.

Question 7.11.* For each of the weighted voting systems in Question 7.4, use Definition 7.10 to determine which voters are dictators, which are dummies, and which have veto power.

Question 7.12.

(a) In a yes/no voting system, does every dictator have veto power? Is every voter with veto power a dictator? Give a convincing argument or example to justify each of your answers.

(b) Can a yes/no voting system have more than one dictator? Can a yes/no system have more than one voter with veto power? Give a convincing argument or example to justify each of your answers.

Swap Robustness

So far in this chapter, we have been presented with voting systems that we knew were weighted. But what if we were faced with a yes/no voting system in which it was clear that the voters were not all treated equally, but not clear whether we could find weights and a quota to view the system as a weighted system? To consider this scenario, let's look at a couple of examples.

Question 7.13.* The United Nations Security Council consists of fifteen representatives—one from each of fifteen different countries. Five representatives (from China, France, Russia, the United Kingdom, and the United States) are considered permanent members, and the remaining ten representatives change from year to year, with nonpermanent members serving two-year terms. Passage of a motion in the council requires votes in favor from all five permanent members and at least four nonpermanent members.

(a) Can the voting system used to make decisions on motions in the U.N. Security Council be represented as a weighted system? If so, find weights and a quota for the system. Otherwise, explain why it cannot.

(b) In the voting system used to make decisions on motions in the U.N. Security Council, are any of the fifteen representatives dictators or dummies? If so, which ones? Do any of them have veto power? If so, which ones?

Question 7.14. The mystical land of Psykozia is divided into four states: Bliss, Confusion, Disarray, and Ignorance. The federal government of Psykozia is similar to that of the United States. It is composed of:

- a four-member Senate, with exactly one senator from each state;
- a five-member House of Representatives, with one representative from each state except the highly populated Bliss, which has two representatives; and
- a president and a vice president.

As in the United States, passage of a federal bill in Psykozia requires majority support in both the House and the Senate, with the vice president casting a tie-breaking vote in the case of a tie in the Senate. The bill is then passed on to the president, who can either sign the bill into law or veto it. A presidential veto can be overridden by a *supermajority* consisting of at least three senators and four representatives.

(a) List all of the different types of winning coalitions that can occur in Psykozia's federal voting system. (Note: We will allow the vice president to be a member of any coalition, even if the makeup of that coalition is such that the vice president's vote is not needed. For instance, we will count the winning coalition containing all eleven members of Psykozia's federal system separately from the

winning coalition containing all members except the vice president, even though the vice president's vote is not needed in the first coalition.)

(b) Do you think Psykozia's federal voting system can be represented as a weighted system? If so, try to find weights and a quota for the system. Otherwise, explain why the system cannot be represented as a weighted system.

(c) Are any members of Psykozia's federal voting system dictators or dummies? If so, which ones? Do any of them have veto power? If so, which ones?

Questions 7.13 and 7.14 are similar, but you probably found the latter to be more challenging. That's because Question 7.13 can be answered by simply finding the right weights and quota. It might take some trial and error, but the task is doable. Question 7.14 is another story. It's a pretty safe bet that you weren't able to find weights and a quota that would work to represent Psykozia's federal system as a weighted voting system. That's because Psykozia's federal system *isn't* a weighted voting system.

But how would we prove this? The direct approach seems pretty challenging—after all, we would have to explain why it is impossible to find weights and a quota that are compatible with the description given in Question 7.14. We've seen (for instance, in Chapters 5 and 6) what these types of impossibility arguments can look like and how involved they can be. So perhaps we should consider some other options.

One possibility would be to reason indirectly. For example, our argument could look something like this:

- All mammals are warm-blooded.
- My pet iguana is not warm-blooded.
- Therefore, my pet iguana is not a mammal.

Of course, we'd want to replace the words *mammal*, *iguana*, and *warm-blooded* with other words (like *weighted voting system* and *Psykozia's federal voting system*) that would make sense within the context of our problem. But the idea would be the same.

Since we're interested in determining whether Psykozia's federal voting system can be represented as a weighted system, a good first step would be to identify a feature that all weighted systems must possess—like how we identified above that all mammals have the feature of being warm-blooded. We could then try to explain why Psykozia's federal system does not have this feature. If we were able to do so, then by the same type of reasoning that allowed us to conclude that my pet iguana is not a mammal, we'd be able to conclude that Psykozia's federal system cannot be represented as a weighted system.

The feature of weighted voting systems that we'll use involves looking at the outcomes of "swaps" of voters between winning coalitions.

Definition 7.15.

- Let C_1 and C_2 be any two distinct (but possibly overlapping) coalitions for a yes/no voting system. A one-for-one exchange of a voter from C_1 and a voter from C_2 is called a **swap** between C_1 and C_2. To ensure that the same voter will never appear more than once in a coalition, it is required that neither of the swapped voters belong to both coalitions.
- A yes/no voting system is **swap robust** if every possible swap between winning coalitions always leaves at least one of the coalitions still winning.

Question 7.16.* Consider again the three shareholders of Captain Ahab's Fish & Chips from Warmup 7.1.

(a) List all of the possible swaps between the coalitions {Nicholas} and {Doug, Elisabeth}.

(b) List all of the possible swaps between the coalitions {Doug} and {Doug, Nicholas}.

(c) Suppose the shareholders decide to use a bizarre voting system for which the only winning coalitions are {Nicholas} and {Doug, Elisabeth}. Is this system swap robust? Why or why not?

(d) Suppose now that the shareholders decide to use a new voting system for which the only winning coalitions are {Nicholas}, {Doug, Elisabeth}, {Doug}, and {Elisabeth}. Is this system swap robust? Why or why not?

Question 7.17. Can either of the voting systems from parts (c) and (d) of Question 7.16 be represented as a weighted system? Give a convincing argument to justify your answer.

Let's now investigate more fully how the notion of swap robustness is related to whether or not a yes/no voting system can be viewed as weighted. To do so, we'll consider the effect of swaps on the winning coalitions in an arbitrary weighted voting system.

Question 7.18.* Let V be a weighted voting system, and let C_1 and C_2 be any two winning coalitions for V.

(a) How do the weights of C_1 and C_2 relate to the quota for V?

(b) Suppose a swap is made between C_1 and C_2. How does the sum of the weights of C_1 and C_2 after the swap relate to the sum of the weights of C_1 and C_2 before the swap?

(c) Could both C_1 and C_2 be losing coalitions after the swap? Why or why not?

(d) What do your investigations in parts (a)–(c) allow you to conclude about whether V must be swap robust?

If everything went well in Question 7.18, you discovered that swap robustness is a feature that all weighted voting systems must possess. The following theorem summarizes this fact.

Theorem 7.19. Every weighted voting system must be swap robust.

Question 7.20.* Suppose that for a particular yes/no voting system, you find a swap between two winning coalitions that causes both to become losing coalitions. What can you conclude about the system?

Question 7.21.* Suppose you determine that a particular yes/no voting system is swap robust. Can you conclude that it must be possible to represent the system as a weighted system? Give a convincing argument or example to justify your answer. (Hint: You may find it helpful to look back at some of the previous questions in this section.)

Question 7.22. In light of the last few questions and Theorem 7.19, what can you now conclude about Psykozia's federal voting system? Is it swap robust? Can it be represented as a weighted voting system? Give a convincing argument to justify each of your answers.

Trade Robustness

In the previous section, we saw that every weighted voting system must be swap robust. However, we also saw that the property of being swap robust is not completely equivalent to that of being weighted. As we know from Question 7.21, there might (and, in fact, do) exist yes/no voting systems that are swap robust and yet cannot be represented as weighted systems.

At this point, it would seem natural for us to consider whether we can find a property—perhaps related to or similar to swap robustness—such that any yes/no system satisfying that property could in fact be represented as a weighted system. One possible candidate is the following.

Definition 7.23.

- In a yes/no voting system, an arbitrary exchange of voters (not necessarily one-for-one) among at least two coalitions is called a **trade**.
- A yes/no voting system is **trade robust** if every possible trade among winning coalitions always leaves at least one of the coalitions still winning.

Question 7.24.*

(a) Is every swap a trade? Why or why not?

(b) Is every trade a swap? Why or why not?

(c) If a yes/no voting system is swap robust, must it be trade robust? Why or why not?

(d) If a yes/no voting system is trade robust, must it be swap robust? Why or why not?

As we see from Question 7.24, the property of trade robustness is stronger than that of swap robustness. In other words, it is harder for a yes/no voting system to be trade robust than it is for the system to be swap robust. Thus, although we already know that every weighted voting system must be swap robust, we cannot automatically conclude that every weighted system must be trade robust. This does, however, turn out to be true, and we can prove it using an argument similar to that in Question 7.18.

Question 7.25. Let V be a weighted voting system, and let $C_1, C_2, \ldots, C_n$ be any collection of winning coalitions for V.

(a) How do the weights of $C_1, C_2, \ldots, C_n$ relate to the quota for V?

(b) Suppose a trade is made among $C_1, C_2, \ldots, C_n$. How does the sum of the weights of $C_1, C_2, \ldots, C_n$ after the trade relate to the sum of the weights of $C_1, C_2, \ldots, C_n$ before the trade?

(c) Could $C_1, C_2, \ldots, C_n$ all be losing coalitions after the trade? Why or why not?

(d) What do your investigations in parts (a)–(c) allow you to conclude about whether V must be trade robust?

So we now know that every weighted voting system must be trade robust. But is it also true that every yes/no voting system that is trade robust can be represented as a weighted system? As it turns out, the answer to this question is yes. The following theorem summarizes this important result.

Theorem 7.26. A yes/no voting system can be represented as a weighted system if and only if it is trade robust.

Theorem 7.26 was proved in 1992 by mathematicians Alan Taylor and William Zwicker, who noted that the proof of a similar result was established in 1960 by computer scientist C. C. Elgot. Although we won't look at either of these proofs (they are quite challenging!), we will conclude this chapter with one very interesting application of Theorem 7.26.

Question 7.27. The voting system used to amend the Constitution of Canada requires that proposed amendments be approved by at least seven of the ten Canadian provinces, and that the approving provinces contain at least half of the total population of Canada. The distribution of the population of Canada (according to the 2016 Canadian census) is shown in Table 7.2.

(a) Make a list of all of the minimal winning coalitions in the voting system used to amend the Constitution of Canada. (Assume that the voters in the system are the ten Canadian provinces.)

(b) Is the voting system used to amend the Constitution of Canada swap robust? Give a convincing argument or example to justify your answer.

Province	Percentage
Ontario	38.3%
Quebec	23.2%
British Columbia	13.2%
Alberta	11.6%
Manitoba	3.6%
Saskatchewan	3.1%
Nova Scotia	2.6%
New Brunswick	2.1%
Newfoundland and Labrador	1.5%
Prince Edward Island	0.4%

TABLE 7.2. Population distribution of Canada, 2016

(c) Is the voting system used to amend the Constitution of Canada trade robust? Give a convincing argument or example to justify your answer.

(d) What do your answers to parts (b) and (c) allow you to conclude about the voting system used to amend the Constitution of Canada? Explain.

Questions for Further Study

Question 7.28. In the weighted voting system $[101 : 101, 97, 2]$ from part (a) of Question 7.4, how would the distribution of power in the system be affected if Doug decided to sell one of his shares of stock to each of Nicholas and Elisabeth?

Question 7.29. Is the property of weighted voting systems being isomorphic a *transitive* property? That is, if a weighted voting system V_1 is isomorphic to another weighted system V_2, and V_2 is isomorphic to a third weighted system V_3, does V_1 have to be isomorphic to V_3? Give a convincing argument to justify your answer.

Question 7.30. In Definition 7.8, we stated that two weighted voting systems are isomorphic if they have the exact same winning coalitions. Find at least three other properties that isomorphic weighted voting systems would necessarily have in common.

Question 7.31. In Definition 7.10, we defined a dictator in a yes/no voting system as a voter who is present in every winning coalition and absent from

every losing coalition. Write a different definition of a dictator in a yes/no voting system that involves the idea of minimal winning coalitions.

Question 7.32.

(a) Is it possible for a yes/no voting system to have a dictator but no dummies? Give a convincing argument or example to justify your answer.

(b) Is it possible for a yes/no voting system to have a dummy but no dictator? Give a convincing argument or example to justify your answer.

Question 7.33.

(a) What do you think it would mean for a yes/no voting system to be monotone? Use your understanding of monotonicity from previous chapters to write a precise definition.

(b) According to your definition from part (a), is every weighted voting system monotone? Give a convincing argument or example to justify your answer.

(c) Suppose that in our definition of a weighted voting system, we had allowed weights to be zero or negative. Would this change your answer to part (b)?

(d) Find a yes/no voting system that is monotone and swap robust, but that cannot be represented as a weighted voting system.

Question 7.34. Write short biographies of Alan Taylor and William Zwicker, including their current academic positions and anything you can find about their views on voting theory and social choice.

Question 7.35.

(a) Research the voting system used for enacting federal laws in the United States, including the exact circumstances under which a bill can be passed. Write a detailed summary of your findings, including descriptions of the minimal winning coalitions for the system.

(b) Can the United States' federal voting system be represented as a weighted voting system? Give a convincing argument or example to justify your answer.

Question 7.36. In the United States' federal voting system (like that of Psykozia from Question 7.14), the president can veto a bill, effectively sending it back for further action or death in the Senate and House. Find out what further action must be taken in the Senate and House to avoid the death of a vetoed bill. Also, according to the definitions presented in this chapter, does the president in the United States' federal voting system have veto power? Why or why not?

Question 7.37. Without using either of the ideas of swap robustness and trade robustness, write a convincing argument that Psykozia's federal voting system (from Question 7.14) cannot be represented as a weighted system.

Question 7.38. Investigate the voting systems used over the years in the Council of the European Union, and write a detailed summary of your findings.

Question 7.39. Sybil and Joanne are adjusting to life with their young son, Curtis. They are quickly finding that leaving the house is not as simple as it used to be. If Sybil and Joanne want to go somewhere without hiring a babysitter, then Curtis must in some sense approve of the trip. For example, if Curtis is sleeping, ill, or otherwise unable to leave the house, then at least one of Sybil or Joanne must stay home with him. Of course, Curtis is also incapable of leaving the house by himself; if he wants to go to his favorite toy store, he must have the approval (or, perhaps more appropriately, cooperation) of at least one of his parents. Consider this family's process of deciding whether or not to go somewhere as a yes/no voting system, with each member of the family being one of the voters.

(a) List all of the winning coalitions for this system.

(b) Are there any dictators, dummies, or voters with veto power in this system?

(c) Is this system weighted? Why or why not?

Question 7.40. Fred, Wilma, Pebbles, and Bam-Bam are trying to decide whether they should stay in and order pizza for dinner or go out to the Dino Rock cafe. To decide this issue, they use a voting system that has the following winning coalitions:

$$\{F, W\}, \{W, P\}, \{P, B\}, \{F, B\}, \{F, W, P\},$$
$$\{F, W, B\}, \{W, P, B\}, \{F, P, B\}, \{F, W, P, B\}$$

(a) Is this voting system swap robust? Why or why not?

(b) What does your answer to part (a) allow you to conclude about whether the system is weighted or not?

(c) Suppose that, in addition to the winning coalitions specified earlier, $\{W, B\}$ and $\{F, P\}$ were also winning coalitions. Would the system then be swap robust?

(d) With the addition of these two winning coalitions, would your answer to part (c) allow you to conclude that the system can be represented as a weighted system?

(e) Is the revised system from part (c) weighted? If so, find weights and a quota for it. Otherwise, explain why it cannot be represented as a weighted system.

Answers to Starred Questions

7.3. In a weighted system for Warmup 7.1, the voters would be Doug, Nicholas, and Elisabeth. The most obvious choices for the weights would be 101 for Doug, 97 for Nicholas, and 2 for Elisabeth. Since the sum of the weights in the system is 200, a reasonable choice for the quota would be 101.

7.4. (a) Any combination of voters that includes Doug could cause the motion to pass by voting in favor of it. With this quota, the outcome of the voting on any motion would always be identical to however Doug voted.

(b) Any combination of voters that includes Doug and at least one other voter could cause the motion to pass by voting in favor of it. With this quota, Doug could not singlehandedly force a motion to pass by voting in favor of it, but he could force a motion to fail by voting against it.

(c) Both Doug and Nicholas would have to vote in favor of a motion in order for it to pass, and it wouldn't matter how Elisabeth voted. With this quota, Doug and Nicholas would be equally powerful, and Elisabeth would be completely powerless.

7.6. (a) The winning coalitions are {Doug}, {Doug, Nicholas}, {Doug, Elisabeth}, and {Doug, Nicholas, Elisabeth}. Only {Doug} is minimal.

(b) The winning coalitions are {Doug, Nicholas}, {Doug, Elisabeth}, and {Doug, Nicholas, Elisabeth}. Both {Doug, Nicholas} and {Doug, Elisabeth} are minimal.

7.9. (a) The winning coalitions are $\{v_1, v_2\}$ and $\{v_1, v_2, v_3\}$.

(b) The winning coalitions are $\{v_1, v_2\}$, $\{v_1, v_3\}$, and $\{v_1, v_2, v_3\}$.

(c) This system is isomorphic to the one in part (a).

7.11. In part (a), Doug is a dictator and has veto power, and Nicholas and Elisabeth are both dummies.

7.13. The system can be represented as a weighted system. You may be able to find weights and a quota by trial and error. Or you may want to proceed in a more systematic manner; one way to do so is to suppose each nonpermanent member has a single vote, each permanent member has x votes, and the quota is q. Then since votes in favor of a motion from all five permanent members and just four nonpermanent members should cause the motion to pass, $q \leq 5x+4$. And since votes in favor from just four permanent members and all ten nonpermanent members should cause the motion to fail, $4x + 10 < q$. What can you conclude about x by combining these two inequalities? And then what can you conclude about q?

7.16. (a) There are two possible swaps: Nicholas and Doug, and Nicholas and Elisabeth.

(b) There are no possible swaps.

(c) This system is not swap robust. Swapping Nicholas and Doug leaves both coalitions losing.

7.18. (a) The weights of C_1 and C_2 must both be greater than or equal to the quota for V. Note also that the sum of their weights must then be greater than or equal to twice the quota for V.

(b) The sum of the weights of C_1 and C_2 after the swap would be identical to the sum of the weights of C_1 and C_2 before the swap, since the two coalitions together contain the same voters after the swap as before.

(c) If C_1 and C_2 were both losing coalitions after the swap, then the sum of their weights after the swap would have to be less than twice the quota for V. This contradicts the answers to parts (a) and (b).

7.20. You could conclude that the system cannot be represented as a weighted system.

7.21. You could not conclude that it must be possible to represent the system as a weighted system. (You could also not conclude that it must be *impossible* to represent the system as a weighted system.)

7.24. (a) A swap is a special kind of trade.

(b) There are lots of trades that are not swaps. (Can you give an example of one?)

(c) A yes/no voting system that is swap robust need not be trade robust.

(d) A yes/no voting system that is trade robust must be swap robust.

Chapter 8

Calculating Corruption

Power tends to corrupt, and absolute power corrupts absolutely.
– Lord Acton, British historian of the late nineteenth and early twentieth centuries.
Power corrupts; absolute power is really neat!
– Donald Regan, White House Chief of Staff, 1985–1987.

Focus Questions

In this chapter, we'll explore the following questions:

- What are some ways of measuring the power held by each of the voters in a yes/no voting system?
- In a yes/no voting system, what is the difference between critical voters and pivotal voters?
- What are the differences between the Banzhaf power index and the Shapley-Shubik power index?
- What is combinatorics? How can the tools of combinatorics be used to calculate power indices?

Warmup 8.1. Consider again the three shareholders of Captain Ahab's Fish & Chips (from Warmup 7.1) and the decision they must make about the fate of their vice president of marketing, Deanne Boomhauer. For reference, the number of shares of stock held by each shareholder is shown again in Table 8.1. Suppose that after much debate, Doug, Nicholas, and Elisabeth agree to adopt a $[103 : 101, 97, 2]$ weighted voting system to make their final decision on Boomhauer.

Shareholder	Shares
Doug	101
Nicholas	97
Elisabeth	2

TABLE 8.1. The shareholders of Captain Ahab's Fish & Chips

(a) In this weighted voting system, note that Nicholas's weight is more than 48 times Elisabeth's. Does this mean that Nicholas is more than 48 times as powerful as Elisabeth? If not, then exactly how much more powerful is Nicholas than Elisabeth?

(b) In this system, does Doug have more power than Nicholas? If so, how much more?

(c) What percentage of the total power in the system does Doug have? What about Nicholas? Elisabeth?

In Warmup 8.1, you were asked some fairly specific questions about the amount of power held by each of the voters in a weighted voting system. These types of questions arise naturally whenever we are dealing with voting systems in which the voters are not all treated equally.

How you answered Warmup 8.1 probably depends on how you interpreted the word *power*. What does it mean to be "powerful" in the context of a democratic process? And what does it mean to say that one person in such a process has more power than another? Can we quantify this notion of power in order to make meaningful comparisons between the participants in a political system? If so, how?

Our goal in this chapter is to develop some mathematically precise ways of answering questions such as these. To do so, we'll investigate two different methods for measuring the power held by the voters in a yes/no voting system. Each of these methods is called a **power index** because it assigns to each voter some numerical measure of that voter's power. In addition to the power indices themselves, we'll also discover some new mathematical tools that will help us more easily calculate the distribution of power for a number of interesting examples.

The Banzhaf Power Index

The first power index we'll consider was proposed in 1965 by John F. Banzhaf III. Throughout his career as a lawyer and professor of law, Banzhaf specialized in public interest law and public health. He is most well known for his role in a series of lawsuits against the tobacco and fast-food industries, which you may have read about or seen on television.

Banzhaf's views on how power is distributed among voters in yes/no voting systems were based on his belief that a particular voter is more powerful than another if that voter's membership in winning coalitions is more frequently essential, or *critical*, to keeping the coalitions from being losing coalitions. The specifics of the Banzhaf index are given in the next definition, and the questions that follow will help you better understand exactly how the index works.

Definition 8.2.

- A voter in a winning coalition is said to be **critical** if the voter's withdrawal from the coalition would cause it to become a losing coalition.
- The **Banzhaf power** of a voter in a yes/no voting system is the number of winning coalitions in which the voter is critical.
- The **total Banzhaf power** of a yes/no voting system is the sum of the Banzhaf powers of all of the voters in the system.
- The **Banzhaf index** of a voter in a yes/no voting system is the Banzhaf power of the voter divided by the total Banzhaf power of the system.

Question 8.3.* Consider the weighted voting system from Warmup 8.1.

(a) Make a list of all of the winning coalitions for the system.

(b) In how many of the winning coalitions from part (a) is Doug critical? In how many is Nicholas critical? What about Elisabeth?

(c) Using your answers to part (b), determine the Banzhaf power of each of the three voters in the system. Then calculate the total Banzhaf power of the system.

(d) Using your answer to part (c), calculate the Banzhaf index of each of the three voters in the system.

Question 8.4. How do your answers to parts (c) and (d) of Question 8.3 compare with your answer to Warmup 8.1?

Question 8.5. Which do you think would be more useful to know: the Banzhaf power of a voter or the Banzhaf index of a voter? Explain.

Question 8.6.

(a) What would the Banzhaf index of a dictator be? What about a dummy? Clearly explain your answers using the terms given in Definition 8.2.

(b) What, if anything, can you say about the Banzhaf index of a voter who has veto power? Give a convincing argument to justify your answer.

Now that we understand the basic idea behind the Banzhaf index, we should be ready to look at a slightly more complicated example. In the next

question, we'll investigate the actual situation that prompted Banzhaf to develop his index in the first place.

Question 8.7.* The Board of Supervisors in Nassau County, New York used a weighted voting system that gave representation to each of six districts in the county according to their relative populations. In 1965, a total of 115 votes were allocated to the districts, as shown in Table 8.2. For a motion to be passed, a simple majority of the total number of votes was required, and thus the quota for the system was 58.

District	Votes
Hempstead 1	31
Hempstead 2	31
Oyster Bay	28
North Hempstead	21
Long Beach	2
Glen Cove	2

TABLE 8.2. Nassau County Board of Supervisors, 1965

(a) In a series of lawsuits, Banzhaf successfully argued that all of the power in the board was equally distributed among the three largest districts. Without actually calculating the Banzhaf power or index of any of the districts, explain why this was in fact true.

(b) Make a list of all of the winning coalitions for the system.

(c) For each of the winning coalitions from part (b), identify all of the voters that are critical to the coalition.

(d) Using your answer to part (c), determine the Banzhaf index of each of the six districts in the system.

(e) Does your answer to part (d) support Banzhaf's claim that all of the power in the board was equally distributed among the three largest districts? Explain.

Question 8.8.* As a result of Banzhaf's lawsuits, the allocation of votes in the Nassau County Board of Supervisors was changed, and actually changed several times before a federal judge declared the board unconstitutional in 1994. The final allocation of votes in the board in 1994 yielded the weighted voting system $[65 : 30, 28, 22, 15, 7, 6]$, with the districts listed in the same order as they are listed in Table 8.2.

(a) Following the same steps as in parts (b)–(d) of Question 8.7, determine the Banzhaf index of each of the six districts in the 1994 system.

(b) In the 1994 system, what percentage of the total power did each of the six districts have? What percentage of the total power did

each of the six districts have in the 1965 system? Does the 1994 system seem more reasonable than the 1965 system?

(c) Suppose that in 1994, a little less than 2% of the population of Nassau County lived in the Glen Cove district. If this were true, then do you think it was acceptable that Glen Cove was given $\frac{6}{108} = 5.56\%$ of the votes in the Nassau County Board of Supervisors in 1994? Why or why not?

The Shapley-Shubik Power Index

The second power index we'll consider is named for economists Lloyd Shapley and Martin Shubik. Shapley and Shubik first proposed their index in 1954, eleven years before Banzhaf proposed his. We looked at the Banzhaf index first though, because the Shapley-Shubik index is somewhat harder to calculate, and the mathematics behind it is slightly more sophisticated.

Shapley and Shubik's views on how power is distributed among voters in yes/no voting systems were based on the idea of *pivotal* voters instead of *critical* voters. Shapley and Shubik believed that coalitions in voting systems were formed sequentially, with some voter joining first, a different voter joining second, another third, and so on. And when it is assumed that members join a winning coalition in some order, it makes sense to talk about which voter first gives the coalition enough total weight to make it a winning coalition. It is this unique voter in an ordered coalition that we will call the *pivotal voter* for the coalition.[1] This terminology is made more precise in the following definition.

Definition 8.9.

- For some arrangement (ordering) of all of the voters in a yes/no voting system, we say that a voter v is **pivotal** if both of the following conditions hold:
  - If each of the voters before v in the arrangement votes for a motion to pass, and v and all of the voters after v vote for the motion to fail, then the motion will fail.
  - If v and all of the voters before v in the arrangement vote for a motion to pass, and all of the voters after v vote for the motion to fail, then the motion will pass.
- The **Shapley-Shubik power** of a voter in a yes/no voting system is the number of arrangements of all of the voters in the system in which the voter is pivotal.
- The **total Shapley-Shubik power** of a yes/no voting system is the total number of arrangements of all of the voters in the system.

[11]Note that this definition is consistent with how we used the term *pivotal voter* in our proof of Arrow's Theorem in Chapter 5.

- The **Shapley-Shubik index** of a voter in a yes/no voting system is the Shapley-Shubik power of the voter divided by the total Shapley-Shubik power of the system.

You might find it interesting that the definition of the Shapley-Shubik index never actually mentions winning coalitions. This is because when we want to calculate a Shapley-Shubik index, we typically just consider every possible arrangement of all of the voters in the system, and then identify the unique pivotal voter in each arrangement. We can do this because every sequentially formed coalition will eventually go from losing to winning as voters are added. Thus, by considering all possible arrangements of all of the voters, we will eventually discover all of the (ordered) winning coalitions. Proceeding in this manner is helpful for a number of reasons, one of which is the fact that we can very easily determine the total number of arrangements of all of the voters in a system by using a simple formula that we will discover in a moment.

But first, an example: Let's find the Shapley-Shubik indices of the three shareholders of Captain Ahab's Fish & Chips.

Question 8.10.*

(a) Make a list of every possible arrangement of all of the voters in the weighted voting system from Warmup 8.1.

(b) Identify the pivotal voter in each of the arrangements from part (a).

(c) Using your answer to part (b), determine the Shapley-Shubik index of each of the voters in the weighted voting system from Warmup 8.1.

Question 8.11. Compare your answer to part (c) of Question 8.10 with your answer to part (d) of Question 8.3.

(a) Should these two answers be the same? Are they the same? Explain any discrepancies between the two, and discuss whether you think it's reasonable for these discrepancies to exist.

(b) Which index do you think better represents the actual distribution of power in the weighted voting system from Warmup 8.1? Clearly explain your answer.

Question 8.12.

(a) What would the Shapley-Shubik index of a dictator be? What about a dummy? Clearly explain your answers using the terms given in Definition 8.9.

(b) What, if anything, can you say about the Shapley-Shubik index of a voter who has veto power? Give a convincing argument to justify your answer.

As we saw in Question 8.10, not only are the calculations involved in finding Shapley-Shubik indices different from those involved in finding Banzhaf indices, but the results could be different too. This is analogous to what we saw in previous chapters when we looked at the outcomes produced by different voting systems for elections with more than two candidates.

There are some other significant differences between the two indices as well. For one thing, the Banzhaf power of an individual voter is generally easier to calculate than the Shapley-Shubik power. However, the total Banzhaf power of a system is generally more difficult to calculate than the total Shapley-Shubik power. To find the total Banzhaf power of a system, we must find the Banzhaf power of each of the voters in the system. But to find the total Shapley-Shubik power of a system, we need only determine the total number of arrangements of all of the voters in the system. The following question suggests how we might go about calculating this total number.

Question 8.13.*

(a) How many different arrangements of two voters are possible?

(b) How many different arrangements of three voters are possible?

(c) How many different arrangements of four voters are possible? (Hint: You could write out each of these arrangements by hand, but it may be easier to just note that each arrangement of four voters can be formed by inserting the fourth voter into some already formed arrangement of the first three voters.)

(d) How many different arrangements of five voters are possible? (Hint: Use your answer to part (c) and the hint given there.)

(e) Do you see a pattern yet? Based on your answers to parts (a)–(d), how many different arrangements of six voters are possible? What about seven voters? Eight voters? n voters (where n just represents some arbitrary number of voters)?

The quantities you calculated in Question 8.13 are called *factorials*. For a whole number n, we write "n-factorial" as $n!$, and we define this quantity using the formula in the answer to part (e) of Question 8.13, which is given at the end of the chapter. (Note: You might want to jot down the formula for $n!$ in the margin here. We'll be using factorials quite a bit in our next round of calculations, so it will be convenient for you to have this formula at your fingertips.)

Now that we've seen how easy it is to find the total Shapley-Shubik power of a voting system, let's investigate the calculations involved in finding Shapley-Shubik indices for a more complicated example. We'll consider again the weighted voting system used by the 1965 Nassau County Board of Supervisors, which we first looked at in Question 8.7. Since there are six

voters in the system (the six districts listed in Table 8.2), the total Shapley-Shubik power of the system is $6! = 720$. So all we need to do now is make a list of the 720 possible arrangements of the voters in the system and identify the pivotal voter in each one. Right?

Okay, maybe not. What we really need is a way to simplify this process, or at least break it down into smaller, more easily digestible chunks. The next question suggests one way to do just that.

Question 8.14.* Consider the weighted voting system from Question 8.7, with voters and weights as shown in Table 8.2 and a quota of 58.

(a) Suppose that in a particular arrangement of the six districts, the ordering of the three largest districts is Hempstead 2, followed in some later position by Oyster Bay, followed in some later position by Hempstead 1. Which of the six districts would be pivotal in such an arrangement?

(b) Does your answer to part (a) depend on where the three smallest districts are placed within the arrangement? Explain.

(c) Suppose that the ordering of the three smallest districts is North Hempstead, followed in some later position by Long Beach, followed in some later position by Glen Cove. How many different arrangements of all six districts would be consistent with this ordering *and* the ordering of the three largest districts specified in part (a)? (Hint: Each such arrangement is completely determined by which positions the largest three districts occupy. How many ways are there to choose these three positions?)

(d) Would your answer to part (c) be different if a different ordering of the three smallest districts were assumed?

(e) How many different orderings of the three smallest districts are possible?

(f) Using your answers to parts (b)–(e), determine the total number of arrangements of all six districts that have the three largest districts in the order specified in part (a).

(g) Would your calculations from parts (a)–(f) be different if a different ordering of the three largest districts were assumed?

(h) Using your answers to parts (a)–(g), determine the Shapley-Shubik index of each of the six districts in the system. Show all of your work, and clearly explain your reasoning.

Banzhaf Power in Psykozia

In the remainder of this chapter, we're going to apply what we've learned about the Banzhaf and Shapley-Shubik power indices to Psykozia's federal

voting system, which we first considered in Question 7.14. Our main interest in looking at this system is due to its similarities to the United States' federal voting system. As such, the calculations we're about to do will give you a taste of what would be involved in determining the distribution of power within the United States' system.

Before we dive back into Psykozia, you should know that some of the calculations we'll need to do are quite complicated and involved. If you are a person who likes quick answers, the next several pages will be an exercise in patience and perseverance for you. With that said, know that what we're about to do isn't rocket science; it actually all boils down to multiplication and addition, which you definitely know how to do. And as we've seen before, the trick here will be to proceed in a careful and systematic way.

Now let's get started. Recall that in Psykozia's federal voting system, there are four different types of voters: senators, representatives, the president, and the vice president. We'll begin by calculating the Banzhaf index of one of the senators, whom we'll call S.

Question 8.15.* Recall that in Psykozia's federal voting system, there are four senators and five representatives.

(a) In how many different ways can a coalition consisting of S and two other senators be selected? Clearly explain your answer.

(b) In how many different ways can a coalition consisting of three representatives be selected? Clearly explain your answer.

(c) In how many different ways can a coalition consisting of S, two other senators, and three representatives be selected? Clearly explain your answer.

Hopefully, Question 8.15 warmed you up for the task of calculating the Banzhaf power of S. We'll do this in the next question.

Question 8.16.*

(a) Describe all of the different types of winning coalitions in which S would be critical. (Hint: These coalitions fall into 10 distinct categories, 3 that contain S and one other senator, and 7 that contain S and two other senators.)

(b) For each of the 10 different types of winning coalitions that you described in part (a), count the number of different ways in which that particular type of winning coalition could be formed. (Hint: Use the same kind of reasoning as you used in Question 8.15.)

(c) Using your answers to parts (a) and (b), determine the Banzhaf power of S.

The calculations required for Question 8.16 were somewhat tedious, but like many other things, they get easier once you've done a few. So let's

continue by calculating the Banzhaf power of each of the other voters in the system.

Question 8.17.

(a) Using the same kind of reasoning as you used in Question 8.16, determine the Banzhaf power of an individual representative in Psykozia's federal voting system. (Hint: You will need to consider nine different types of winning coalitions here.)

(b) Determine the Banzhaf power of the president and the vice president in Psykozia's federal voting system.

Question 8.18.*

(a) Using your answers to Questions 8.16 and 8.17, find the Banzhaf index of each of the voters in Psykozia's federal voting system. (Hint: Don't forget that there are four senators and five representatives.)

(b) According to the Banzhaf indices you calculated in part (a), what percentage of the total power in Psykozia's federal voting system is held by the Senate? What about the House of Representatives? The president? The vice president?

(c) Do you think your answers to parts (a) and (b) seem reasonable? Do you think anyone in the system has more power or less power than they deserve, or does the distribution of power seem about right? Explain.

That was a lot of work, but we were finally able to find the Banzhaf index of each of the voters in Psykozia's federal voting system. In just a moment, we'll also find the Shapley-Shubik index of each of these voters. But before we do so, let's take a few minutes to formalize some of the mathematical ideas we just used.

A Splash of Combinatorics

You may not have been aware of it, but when you answered the questions in the previous section, you were dipping your toe into an area of mathematics known as *combinatorics*. As you might guess from your answers to those questions, combinatorics focuses on problems that involve counting objects or combinations of objects, usually in a very precise and systematic way. Calculating Banzhaf indices often involves counting the number of different ways in which a certain set of objects can be chosen. For instance, in part (b) of Question 8.15, we needed to count the number of different ways in which we could choose three of the five representatives in Psykozia's federal voting system. With a little bit of thought, it's not too hard to see that there are exactly 10 ways to do this.

But what if the situation was more complicated? For instance, what if we needed to count the number of different ways to choose 51 out of the 100

senators in the United States' federal voting system? This is exactly the kind of question that the tools of combinatorics can help us answer. Let's begin with a bit of notation.

Definition 8.19. The number of different ways to choose k objects out of a collection of n objects is denoted by the symbol $\binom{n}{k}$, which is read "n choose k."

Question 8.20.* Find the value of each of the following quantities. (Note: Although you may already know a formula for $\binom{n}{k}$, you shouldn't need to use it for this question; just think about what each quantity represents.)

(a) $\binom{5}{0}$

(b) $\binom{5}{1}$

(c) $\binom{5}{2}$

(d) $\binom{5}{3}$

(e) $\binom{5}{4}$

(f) $\binom{5}{5}$

(g) $\binom{5}{1} + \binom{5}{2}$

(h) $\binom{6}{2}$

If you look carefully at your answers to Question 8.20, you'll see evidence of a number of useful properties. For example, it must be the case that $\binom{5}{0} = 1$, since there is exactly one way to choose zero objects out of a collection of five objects. (The one way is to not choose any of the five objects.) But the same reasoning would also apply if we were considering a collection of six or eight or a hundred objects. Thus, we have the following result.

Theorem 8.21. *For any value of n,* $\binom{n}{0} = 1$.

As simple as it is, the previous theorem is one of many useful facts that will help us to find $\binom{n}{k}$ for any values of n and k we desire. In the next

two questions, we'll identify some of the other properties that will also be helpful in this regard.

Question 8.22.* What is the relationship between $\binom{n}{k}$ and $\binom{n}{n-k}$? Give a convincing argument to explain why this relationship will hold for all possible values of n and k. (Hint: It may be helpful to look back at your answers to Question 8.20.)

Question 8.23.* Fill in the blanks to make each of the following statements true. Then give a convincing argument to explain why each statement is true. (Hint: You may want to look at some examples first, especially for part (c).)

(a) For any value of n, $\binom{n}{1} =$ _____.

(b) For any value of n, $\binom{n}{n} =$ _____.

(c) For any values of n and k, $\binom{n}{k-1} + \binom{n}{k} = \binom{\quad n+1 \quad}{\quad\quad}$.

Now let's put all of these facts together. In doing so, we'll catch a glimpse of a famed mathematical tool known as Pascal's triangle, named after 17th-century mathematician Blaise Pascal. Pascal's triangle contains the various values of $\binom{n}{k}$ in a triangular array, the top four rows of which are shown in Figure 8.1. The triangle continues on forever, following the same pattern, with each row containing one more entry than the row above it.

$$\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\
 & & & \vdots & & &
\end{array}$$

FIGURE 8.1. Pascal's triangle

If we label the rows of Pascal's triangle starting at the top with zero, and also label the entries in each row starting at the left with zero, then the

kth entry in the nth row is $\binom{n}{k}$. The top four rows, with actual numerical values for $\binom{n}{k}$, are shown in Figure 8.2.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \\ & & & \vdots & & & \end{array}$$

FIGURE 8.2. Pascal's triangle

Question 8.24.

(a) What number appears at the beginning and at the end of each row of Pascal's triangle? Which property of $\binom{n}{k}$ lets you conclude this?

(b) Explain how the numbers in any row of Pascal's triangle can always be found from the numbers in the row above it. (Hint: You may want to look back at your answer to part (c) of Question 8.23.)

Question 8.25.*

(a) Using your answers to Question 8.24, write the next four rows of Pascal's triangle (after the last row shown in Figure 8.2).

(b) Use your answer to part (a) to find $\binom{7}{3}$.

(c) Suppose you needed to find $\binom{9}{6}$. How could you do it?

Shapley-Shubik Power in Psykozia

Now that we know some combinatorics, we're finally ready to calculate the Shapley-Shubik indices of the voters in Psykozia's federal voting system. Just like we did when we considered Banzhaf power in Psykozia, we'll begin here by calculating the Shapley-Shubik index of one of the four senators, whom we'll again call S.

Question 8.26.*

(a) Suppose all of the voters in Psykozia's federal voting system are arranged in some order so that S is preceded by exactly two other senators and three representatives. Is S pivotal in this type of arrangement?

(b) Suppose all of the voters in Psykozia's federal voting system are arranged in some order so that S is preceded by exactly two other senators and four representatives. Is S pivotal in this type of arrangement?

(c) In how many different ways can the six voters preceding S in part (b) be selected?

(d) Once the six voters preceding S in part (b) are selected, in how many different ways can these six voters be arranged in some order?

(e) Once the six voters preceding S in part (b) are selected and arranged, how many voters will be left to follow S in an arrangement of all of the voters in the system?

(f) In how many different ways can the voters following S in part (e) be arranged in some order?

(g) Based on your answers to parts (c)–(f), in how many different ways can all of the voters in Psykozia's federal voting system be arranged so that S is preceded by exactly two other senators and four representatives?

Question 8.27.*

(a) Describe all of the different types of arrangements of the voters in Psykozia's federal voting system in which S would be pivotal. (Hint: These arrangements fall into ten distinct categories, one of which is identified in part (b) of Question 8.26.)

(b) Calculate the number of different ways in which each type of arrangement from part (a) could be formed. (Hint: Use the same kind of reasoning as you used in Question 8.26.)

(c) Using your answers to parts (a) and (b), determine the Shapley-Shubik index of S.

Question 8.28.* Using the same kind of reasoning as you used in Question 8.27, determine the Shapley-Shubik index of an individual representative in Psykozia's federal voting system. Then do the same for the president and the vice president.

Question 8.29.

(a) Does anything about the Shapley-Shubik indices of the voters in Psykozia's federal voting system strike you as being strange or unusual? Explain.

(b) Compare the Banzhaf and Shapley-Shubik indices of the voters in Psykozia's federal voting system. Do you think these two indices will always give such similar results? If so, explain why. Otherwise, give an example of a system for which the results of the two indices would be significantly different.

Question 8.30. Write a page or two comparing Banzhaf's ideas on how power is distributed in yes/no voting systems with Shapley and Shubik's. Include in your summary answers to at least the following questions:

- Which index do you think more accurately represents the distribution of power within a system: the Banzhaf index or the Shapley-Shubik index? Clearly explain your reasoning.
- Do you think that the index you specified above will be better all of the time, most of the time, or just some of the time? Explain.
- Which do you think play a more important role in yes/no voting systems: critical voters or pivotal voters? Explain.

Questions for Further Study

Question 8.31. Using the idea of a critical voter, write new definitions for the terms *dictator*, *veto power*, and *dummy* (originally defined in Definition 7.10). Then do the same thing using the idea of a pivotal voter.

Question 8.32.

(a) What must the sum of the Banzhaf indices of all of the voters in a yes/no voting system be? Give a convincing argument to justify your answer.

(b) Would your answer to part (a) be different for Shapley-Shubik indices than for Banzhaf indices? Why or why not?

Question 8.33. Consider again the weighted voting system from Warm-up 8.1. Calculate the Banzhaf and Shapley-Shubik indices of each of the voters, but this time assume that the quota for the system is 101. Then repeat the calculations for a quota of 105.

Question 8.34.

(a) Find the Banzhaf index of each of the voters in the weighted voting system $[65 : 30, 28, 22, 15, 13]$.

(b) Note that in the weighted voting system used by the Nassau County Board of Supervisors in 1994 (see Question 8.8), if the Long Beach and Glen Cove districts agreed to always vote together, then the voting system used by the board would be equivalent to the system in part (a). With this in mind, use your answers to part (a) and Question 8.8 to answer the following questions:

- If the Long Beach and Glen Cove districts agreed to always vote together, would they have more, less, or the same amount of combined power compared to what they had before the agreement?
- Would the Banzhaf indices of any of the other districts be affected by this agreement?

- Does anything about your answers to parts (a) and (b) strike you as being strange or unusual? Explain.

Question 8.35. Calculate the Shapley-Shubik index of each of the six districts in the weighted voting system used by the Nassau County Board of Supervisors in 1994. (See Question 8.8.) How do the Shapley-Shubik indices compare to the Banzhaf indices that you calculated for the same system in part (a) of Question 8.8? Which index do you think gives a better representation of how the power is truly distributed in the system? Clearly explain your answer.

Question 8.36. Write a short biography of John F. Banzhaf III, including the academic positions he has held and some information about his most notable court cases.

Question 8.37. Write short biographies of Lloyd Shapley and Martin Shubik, including where they met, where they were employed when they came up with the idea for their index, and any other notable contributions they have made both inside and outside of voting theory.

Question 8.38. The quantities we denoted by $\binom{n}{k}$ are often called *binomial coefficients*. Research the meaning of this name, and write a summary of your findings. Include in your summary a description of at least one mathematical application of binomial coefficients outside of voting theory.

Question 8.39. In combinatorics, the quantity $\binom{n}{k}$ is often defined by the following formula:

$$\binom{n}{k} = \frac{n!}{k! \times (n-k)!}.$$

Explain why this definition of $\binom{n}{k}$ is completely consistent with the definition we used in this chapter.

Question 8.40. It is a well-known fact in combinatorics that for any value of n,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$$

Without doing any calculations, explain why this equation is true. (Hint: Explain how the two sides of the equation can be viewed as two different ways of counting the same thing.)

Question 8.41. Recall from Question 7.13 that the voting system used to make decisions on motions in the U.N. Security Council can be viewed as the weighted system $[39 : 7, 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$. Find the Banzhaf and Shapley-Shubik indices of each of the voters in this system.

Question 8.42. In Question 7.27, we considered the voting system used to amend the Constitution of Canada. Find the Banzhaf index of each of the voters in this system (the ten Canadian provinces), and comment on anything about the results that strikes you as being strange or unusual.

Question 8.43. Suppose that in the country of Psykozia, the states of Ignorance and Bliss decide to merge to form a new state, Enlightenment. (So, in this case, ignorance really is bliss!) What corresponding changes do you think should be made to Psykozia's federal voting system? With these changes, would the people of Enlightenment have more, less, or the same amount of combined power compared to what they had before the merger?

Question 8.44. Research the voting system used by the original European Economic Community (EEC), which was established in 1958 as the predecessor to the European Union. Find the Banzhaf and Shapley-Shubik indices of each of the voters in the voting system used by the original EEC. Explain any discrepancies between the two indices, and comment on anything that strikes you as being strange or unusual about how power was distributed within the EEC.

Question 8.45. Find a book or an article that gives the Banzhaf or Shapley-Shubik index of each of the voters in the United States' federal voting system, and write a summary of your findings. Include in your summary answers to at least the following questions: Did the book or article mention how the calculations were done? Do you think the indices accurately represent how the power is truly distributed in the system? Does anything about the indices strike you as being strange or unusual? (Note: There are a total of 537 voters in the system—100 senators, 435 representatives, the president, and the vice president.)

Question 8.46. In addition to the Banzhaf and Shapley-Shubik power indices, there are a number of other methods that can be used to measure how power is distributed within a yes/no voting system. Two of the more common measures are the Deegan-Packel index and the Johnston index. Research both of these indices, and write a detailed summary of your findings. Include in your summary a comparison of the two indices with each other and with the Banzhaf and Shapley-Shubik indices. Also calculate the values of the Deegan-Packel and Johnston indices for each of the six districts in the weighted voting system from Question 8.8.

Question 8.47. Calculate the Banzhaf and Shapley-Shubik indices of each of the voters in the system from Question 7.39.

Question 8.48. Calculate the Banzhaf and Shapley-Shubik indices of each of the voters in the system from Question 7.40.

Question 8.49. Consider a weighted voting system with three voters. For each of the following distributions of power, either find weights and a quota

for which the Shapley-Shubik index would yield the given distribution, or explain why it is impossible to do so.

(a) $1, 0, 0$

(b) $\frac{5}{6}, \frac{1}{6}, 0$

(c) $\frac{4}{6}, \frac{2}{6}, 0$

(d) $\frac{4}{6}, \frac{1}{6}, \frac{1}{6}$

(e) $\frac{3}{6}, \frac{3}{6}, 0$

(f) $\frac{3}{6}, \frac{2}{6}, \frac{1}{6}$

(g) $\frac{2}{6}, \frac{2}{6}, \frac{2}{6}$

Question 8.50. Using the Shapley-Shubik index to measure power, find a weighted voting system with four voters (call them A, B, C, and D) for which:

- A is twice as powerful as B;
- B is twice as powerful as C; and
- C and D have the same amount of power.

Question 8.51. Show that the Banzhaf and Shapley-Shubik indices are always identical when there are only two voters.

Answers to Starred Questions

8.3. (a) The winning coalitions are {Doug, Nicholas}, {Doug, Elisabeth}, and {Doug, Nicholas, Elisabeth}.

(b) Doug is critical in all three winning coalitions. Nicholas and Elisabeth are each critical in only one winning coalition.

(c) The Banzhaf power of Doug is 3, and the Banzhaf power of both Nicholas and Elisabeth is 1. The total Banzhaf power of the system is 5.

(d) The Banzhaf index of Doug is 3/5, and the Banzhaf index of both Nicholas and Elisabeth is 1/5.

8.7. (a) This question is much easier than it seems. Take two of the three largest districts along with zero of the three smallest districts, and see what the weights of the resulting coalitions would be. Then take one of the three largest districts along with all three of the smallest districts, and see what the weights of the resulting coalitions would be.

(b) There are a total of 32 winning coalitions for the system. Of these, 3 have two members, 10 have three members, 12 have four members, 6 have five members, and 1 has six members.

(c) There are a total of 48 critical voters in all of the winning coalitions for the system.

(d) According to your answers to parts (a) and (c), before you even start part (d) you should know what the Banzhaf powers and indices will end up being for each of the six districts.

8.8. (a) There are a total of 23 winning coalitions for the system. Of these, 5 have three members, 11 have four members, 6 have five members, and 1 has six members. There are a total of 52 critical voters in all of the winning coalitions for the system.

(b) In the 1994 system, the percentage of power held by Hempstead 2 was exactly 25%, and the percentage of power held by Glen Cove was 1.92%.

8.10. (a) There are 6 different arrangements of Doug, Nicholas, and Elisabeth. Abbreviating using just the first letter of each name, these 6 arrangements are DNE, DEN, NDE, NED, EDN, and END.

(b) Doug is pivotal in four arrangements. Nicholas and Elisabeth are each pivotal in only one arrangement.

(c) Doug has a Shapley-Shubik index of 4/6, and Nicholas and Elisabeth each have a Shapley-Shubik index of 1/6.

8.13. (a) For two voters, there are only 2 possible arrangements.

(b) For three voters, there are 6 possible arrangements.

(c) For four voters, there are 24 possible arrangements. These arrangements can be found by inserting the fourth voter into each of 4 different places in each of the 6 arrangements from part (b).

(d) For five voters, there are $5 \times 24 = 120$ possible arrangements.

(e) For n voters, there are $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ possible arrangements.

8.14. (a) Oyster Bay would be pivotal.

(b) Oyster Bay would be pivotal regardless of where the three smallest districts were placed within the arrangement.

(c) There are 20 different arrangements that would be consistent with both orderings. If you abbreviate the names of the districts, you should be able to write them all down.

(d) The answer to part (c) would be the same, even if a different ordering of the three smallest districts were assumed.

(e) There are six different orderings of the three smallest districts.

(f) For each ordering of the three smallest districts, there are 20 arrangements of all six districts that have the three largest districts arranged in the order specified in part (a). Since there are six possible orderings of the three smallest districts, there must be $6 \times 20 = 120$ arrangements in total of all six districts that adhere to the ordering of the three largest districts specified in part (a).

(g) If a different ordering of the three largest districts were assumed, then a different district might be pivotal. Other than this, all of the calculations from parts (a)–(f) would remain the same.

(h) The Shapley-Shubik indices for the districts should be the same as the Banzhaf indices you found in part (d) of Question 8.7.

8.15. (a) There are 3 different ways to select a coalition consisting of S and two other senators.

(b) There are 10 different ways to select a coalition consisting of three representatives.

(c) For each of the 3 different ways to select S and two other senators, there are 10 different ways to select three representatives. Thus there are $3 \times 10 = 30$ different ways to select a coalition consisting of S, two other senators, and three representatives.

8.16. (a) The following are a few of the 10 different types of winning coalitions in which S would be critical:

- S, one other senator, the president, the vice president, and four representatives;
- S, two other senators, and four representatives;
- S, two other senators, the president, and three representatives;
- S, two other senators, the vice president, and four representatives.

(b) Consider the first type of coalition listed in the answer to part (a). To form this type of coalition, we must choose one of the three senators besides S (3 possible choices) and four of the five representatives (5 possible combinations). Thus, there are $3 \times 5 = 15$ ways in which this type of coalition could be formed. The same kind of reasoning will work for the other 9 types of winning coalitions in which S would be critical.

(c) You should find that the Banzhaf power of S is 132.

8.18. The Banzhaf indices of the voters in the system are:

- $\frac{132}{1500}$ for each of the senators;

- $\frac{136}{1500}$ for each of the representatives;
- $\frac{196}{1500}$ for the president; and
- $\frac{96}{1500}$ for the vice president.

8.20. (a) 1

(b) 5

(c) 10

(d) 10

(e) 5

(f) 1

(g) 15

(h) 15

8.22. If you're having trouble explaining your answer, consider this: If you choose k objects out of a collection of n objects, how many of the n objects will *not* be chosen?

8.23. (c) For any values of n and k,

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

To see why this is true, suppose you chose k widgets out of a collection of $n+1$ widgets, of which one was red and n blue. If the red widget was among the k that you chose, then how many blue widgets would you have chosen? If the red widget was not among the k that you chose, then how many blue widgets would you have chosen? Now try to extend this reasoning to formulate a general argument.

8.25. (a)

$$\begin{array}{ccccccccc}
 & & & & 1 & 4 & 6 & 4 & 1 \\
 & & & 1 & 5 & 10 & 10 & 5 & 1 \\
 & & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\
 & 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1
\end{array}$$

(b) $\binom{7}{3} = 35$

8.26. (a) S is not pivotal in this type of arrangement.

(b) S is pivotal in this type of arrangement.

(c) There are $\binom{3}{2} \times \binom{5}{4} = 15$ different ways in which these six voters can be selected.

(d) There are $6! = 720$ different ways in which the six selected voters can be arranged.

(e) There will be four voters left over: a senator, a representative, the president, and the vice president.

(f) There are $4! = 24$ different ways in which the four voters that are left can be arranged.

(g) There are $15 \times 720 \times 24 = 259{,}200$ different ways in which all of the voters in the system can be arranged in some order so that S is preceded by exactly two other senators and four representatives.

8.27. The Shapley-Shubik index of S is approximately .087, or 8.7%.

8.28. The Shapley-Shubik index of an individual representative is approximately .091, or 9.1%. The Shapley-Shubik index of the president is approximately .136, or 13.6%, and the Shapley-Shubik index of the vice president is approximately .061, or 6.1%.

Chapter 9

The Ultimate College Experience

I consider it completely unimportant who will vote, or how; but what is extraordinarily important is this—who will count the votes, and how.
– Joseph Stalin

Focus Questions

In this chapter, we'll explore the following questions:

- What is the Electoral College, and how does it work?
- What is the winner-take-all rule, and what are its implications for the Electoral College and U.S. presidential elections?
- Why was the Electoral College created, and why is it still used today?
- Are there any viable alternatives to the Electoral College? If so, what are they?

Warmup 9.1. As you may know, U.S. presidential elections are decided using an institution called the Electoral College. However, suppose for this question that the winner of the 2016 U.S. presidential election was determined using instant runoff based on the popular votes received by each candidate nationwide. These popular vote totals, as reported on the web site of the U.S. Federal Election Commission (`http://www.fec.gov`), are shown in Table 9.1. For convenience, suppose Donald Trump was the second choice of all of the Gary Johnson voters in the election, and Hillary Clinton was the second choice of all of the Jill Stein voters.

Candidate	Popular Votes
Hillary Clinton	65,853,516
Donald Trump	62,984,825
Gary Johnson	4,489,221
Jill Stein	1,457,216
Others	1,884,459

TABLE 9.1. 2016 U.S. presidential election

(a) If the second choice of the "Others" voters had been split as evenly as possible between Clinton and Trump, who would have won this election under instant runoff?

(b) If the second choice of the "Others" voters had been split as closely as possible to 60% for Clinton and 40% for Trump, who would have won this election under instant runoff?

(c) Do you think there is a way to divide the second choice of the "Others" voters between Clinton and Trump so that one of these two candidates would have beaten the other in the final round of instant runoff by a single popular vote? If so, can you find that split (either in percentages or actual numbers of voters)?

As we discussed in Chapter 2, the 2016 U.S. presidential election was hotly contested and provoked a staggering amount of controversy in the political world. Even so, at least mathematically, the 2000 U.S. presidential election was even more controversial. Counting and recounting of the popular votes in Florida in 2000 delayed final certification of the results until a full 36 days after the votes were cast, when the U.S. Supreme Court decided by a 5-4 margin to block further recounts, leaving George W. Bush as the winner of the state and, consequently, the presidency. Bush's certified vote total in Florida was larger than Al Gore's by only 537 out of the almost 6 million total votes that had been cast statewide.

What is more remarkable is that in the 2000 election, Bush was declared the overall winner while receiving more than half a million fewer popular votes than Al Gore nationwide. This bizarre behavior was repeated, and in fact exceeded, in the 2016 election, when Trump was declared the overall winner while receiving almost three million fewer popular votes than Hillary Clinton nationwide. (See Table 9.1.) In this chapter, we'll learn about the system that resulted in these strange outcomes and how it came to be.

The Electoral College

The voting system used by the United States to elect its president is one of the most important examples of its kind, as well as one of the most unusual. No other country uses such a system to elect its head of state. But arguably

the most powerful person on the planet, the President of the United States, is elected using a procedure that effectively operates as a weighted voting system—the famous, and infamous, *Electoral College*.

The Electoral College is really a weighted voting system with a few twists. As mandated in the Constitution, each state casts a number of votes in the Electoral College equal to the number of members of Congress (senators plus representatives) from that state. The only additional votes cast in the Electoral College are three for the District of Columbia (to match the number of electoral votes held by the smallest states). Thus, under the current system, there are 538 electoral votes—435 to represent the members of the House of Representatives, 100 to represent the members of the Senate, and 3 for the District of Columbia.

These 538 electoral votes are actually physically cast by individuals called *electors* who typically meet to vote a few weeks after the November presidential election. Once the votes of these electors are cast, the outcome of the election is determined using majority rule.

Question 9.2.* Consider again the 2016 U.S. presidential election. Suppose that by some amazing twist of fate, Gary Johnson had been able to win the 38 electoral votes from the state of Texas, resulting in the electoral vote totals shown in Table 9.2. Would Johnson's stellar performance in Texas have changed the outcome of the election? If so, how?

Candidate	Electoral Votes
Donald Trump	266
Hillary Clinton	227
Gary Johnson	38
Others	7

TABLE 9.2. A 2016 Electoral College scenario

As you hopefully remembered when you were answering Question 9.2, an election can sometimes result in a tie under majority rule, even when one of the candidates receives more votes than any of the others. Recall that to win an election under majority rule, a candidate must receive *more than half* of the total number of votes. In the Electoral College, this amounts to at least 270 electoral votes, a number you are likely to hear repeated many times during any presidential campaign.

But what if no candidate attains this all-important number of votes? The answer to this question might surprise you. If no candidate receives a majority of the electoral votes, then the electoral (and popular) votes are discarded, and the members of the U.S. House of Representatives vote to determine the winner. This has happened twice throughout U.S. history, in the presidential elections of 1800 and 1824. As we'll see in the next section, there is a very good reason why it has not happened again since then.

The Winner-Take-All Rule

One of the most controversial aspects of the Electoral College is the *winner-take-all* rule, which specifies that all of the electors from each state (with a couple of exceptions we will discuss soon) will vote in the Electoral College for the candidate who won a plurality of the popular votes cast in that state. It is this general rule that makes the Electoral College effectively operate as a weighted voting system, and it is the plurality method that underlies the winner-take-all rule that reveals the first notable deficiency in the system: It is highly manipulable. In every presidential election, there are many voters who would benefit by casting a vote that misrepresents their true preferences. The next question illustrates this phenomenon.

Question 9.3.* Table 9.3 lists the total number of popular votes received in Florida by each of the candidates in the 2000 U.S. presidential election. In addition to these vote totals, assume for this question that the second choice of all of the Browne voters was Gore, and the second choice of the Nader voters was split as closely as possible to 20% for Bush and 80% for Gore.

Candidate	Popular Votes
George W. Bush	2,912,790
Al Gore	2,912,253
Ralph Nader	97,488
Harry Browne	16,415
Others	24,164

TABLE 9.3. 2000 U.S. presidential election in Florida

(a) Suppose that some of the Browne voters (but none of the Nader voters) had decided to switch their votes to their second choice, Gore. How many such voters would have needed to switch their votes in order to change the outcome of the election in Florida (and, consequently, nationwide)?

(b) Suppose that some of the Nader voters (but none of the Browne voters) had decided to switch their votes to their second choice, either Gore or Bush (with 20% of the switched votes going to Bush and 80% to Gore). How many such voters would have needed to switch their votes in order to change the outcome of the election?

(c) In this election, Gore aligned on major issues much more closely with Browne and Nader than with Bush. Despite this, do you think Gore's supporters might have been well-served by running negative campaign ads in Florida attacking Browne or Nader?

How might the arguments presented in such ads have differed from the arguments presented in ads that directly attacked Bush?

Now back to the winner-take-all rule. As we have described, when you cast a popular vote in a presidential election, you are actually casting a vote for who you want *all* of your state's electors to vote for. This is true as long as you don't live in Maine or Nebraska. In both of these states, a single elector is designated to specifically represent each congressional district (two in Maine, three in Nebraska), with two electors left over. The general rule is that an elector representing a particular congressional district will vote in the Electoral College for the candidate who wins a plurality of the popular votes cast in that district, and the two electors left over will vote for the candidate who wins a plurality of the popular votes cast statewide. These rules were adopted by Maine and Nebraska starting with the 1972 and 1992 presidential elections, respectively, although as of this writing, they have each split their electoral votes only once since then, with Barack Obama winning exactly one of Nebraska's electoral votes in 2008 and Donald Trump winning exactly one of Maine's in 2016.

The constitutionality of the winner-take-all rule has been questioned numerous times over the years. This shouldn't be a surprise though, especially considering that there is no federal law requiring individual electors to follow the rule. In several instances the rule has even been violated, including in 2016, when no less than 7 of the 538 electors cast a vote for a different candidate than the one to whom they were pledged. Such electors are sometimes called *faithless* electors. And while faithless electors have the potential to alter the outcome of an election (as some were hoping would happen in 2016), they have never actually done so.

Question 9.4. In the 1992 U.S. presidential election, business magnate H. Ross Perot received almost one fifth of the popular votes, but not a single electoral vote. Explain how the winner-take-all rule contributed to this phenomenon. What other factors were involved in Perot's disappointing Electoral College showing?

As suggested by Question 9.4, the winner-take-all rule has some serious consequences, especially with regard to the viability of third-party candidates. In fact, because of the strong two-party political system that developed in the U.S. during the early 1800s, most presidential elections end up being a choice between only two legitimate contenders. This is true even when there are more than two candidates who receive significant percentages of the popular vote.

So what is the Electoral College good for? Perhaps this question is best answered by considering its history.

Some History

The Electoral College was created and exists today because of a compromise reached at the Constitutional Convention in 1787 between a group of politicians who wanted the president to be directly elected by the entire U.S. population and another group who wanted the president to be chosen by Congress. One purpose of the Electoral College was to place the selection of the president in the hands of a body that was chosen to represent the entire population, but removed from Congress and small enough to make deliberative choices that are possible only among well-informed voters. The winner-take-all approach of the Electoral College developed almost immediately, but only after it became obvious that the American population, though spread out over a very large area that included parts both urban and rural, could be well-informed about the candidates after all.

As you might expect, the Electoral College has evolved over the years, with the most noticeable changes resulting from the addition of states to the Union. Until 1911, when the size of the House of Representatives was fixed by law, the total number of votes in the Electoral College changed with the addition of each new state (and for other reasons that we will discuss in Chapter 11). After 1911, the only changes in the number of electoral votes came in 1959, when Alaska and Hawaii were granted statehood, and in 1961, when the 23rd Amendment to the U.S. Constitution granted the District of Columbia its three electoral votes. Although the total number of electoral votes has remained constant since the 1964 election, their distribution among the states has not. Even though the size of the House of Representatives is currently fixed at 435 members, the number of representatives allocated to each particular state is not fixed. This is why, for example, during the 2004 presidential election, California had 55 electoral votes, instead of the 54 it had during the 2000 election. After the reapportionment of the seats in the House that occurred in 2002 (based on the results of the national census of 2000), California was awarded an additional seat in the House, at the expense of some other state that was forced to give up a seat. In Chapter 11, we will discuss and study the process through which the seats in the House are distributed to the states. For now, however, you may be interested in considering some of the differences in the electoral vote distribution between the 2000 and 2016 elections. For comparison, the numbers of electoral votes held by each state and the District of Columbia in both 2000 and 2016 are shown in Table 9.4.

Question 9.5.* By what percentage did the number of electoral votes held by California increase between the 2000 and 2016 presidential elections?

Question 9.6. Which state experienced the largest percentage increase in its number of electoral votes between the 2000 and 2016 presidential elections? Which state experienced the largest percentage decrease?

State	2000 Votes	2016 Votes	State	2000 Votes	2016 Votes	State	2000 Votes	2016 Votes
AL	9	9	KY	8	8	ND	3	3
AK	3	3	LA	9	8	OH	21	18
AZ	8	11	ME	4	4	OK	8	7
AR	6	6	MD	10	10	OR	7	7
CA	54	55	MA	12	11	PA	23	20
CO	8	9	MI	18	16	RI	4	4
CT	8	7	MN	10	10	SC	8	9
DE	3	3	MS	7	6	SD	3	3
DC	3	3	MO	11	10	TN	11	11
FL	25	29	MT	3	3	TX	32	38
GA	13	16	NE	5	5	UT	5	6
HI	4	4	NV	4	6	VT	3	3
ID	4	4	NH	4	4	VA	13	13
IL	22	20	NJ	15	14	WA	11	12
IN	12	11	NM	5	5	WV	5	5
IA	7	6	NY	33	29	WI	11	10
KS	6	6	NC	14	15	WY	3	3

TABLE 9.4. Electoral votes by state

Power in the Electoral College

Another purpose of the Electoral College was to protect the smaller states by giving each state—regardless of population— two electoral votes for their members of the Senate. For example, these two votes gave the least populous state, Wyoming, which in 2016 had only one seat in the House of Representatives, a 200% boost (2/1) in its electoral vote total. But the most populous state, California, which in 2016 had 53 seats in the House, received only a 3.77% boost (2/53). The end result is that in the smaller states, each elector generally represents a smaller number of people.

Question 9.7.* Using the U.S. Census Bureau's 2016 population total estimate of 585,501 for Wyoming, find the number of people represented by each of this state's electors in the 2016 presidential election.

Question 9.8.

(a) Using the U.S. Census Bureau's 2016 population total estimate of 39,250,017 for California, find the number of people represented by each of this state's electors in the 2016 presidential election.

(b) Briefly discuss how you feel about the fairness of the Electoral College in light of your answers to Question 9.7 and part (a) of this question.

(c) Using your answers to Question 9.7 and part (a) of this question, explain how you might convince someone that it is better for U.S. presidential candidates to campaign in Wyoming than in California.

Despite your answer to part (c) of Question 9.8, you would probably agree that in reality it is much more beneficial for U.S. presidential candidates to campaign in large states like California than in small states like Wyoming. The winner-take-all rule in the Electoral College means that receiving a majority of the popular votes in Wyoming only guarantees a candidate 3 electoral votes, as opposed to the 55 that would currently be guaranteed in California. This actually makes individual voters in California much *more* powerful in presidential elections than voters in Wyoming.

But there is a lot more to the Electoral College story than just the sizes of the states and the numbers of electoral votes they control. For one thing, we learned in Chapter 8 that a voter's power within a weighted voting system is not always directly proportional to the number of votes they control. As we saw then, power is often more accurately described by measures such as the Banzhaf index and the Shapley-Shubik index, each of which takes into account factors other than simply the number of votes held by each voter. But how would we go about calculating these indices for the Electoral College? The next two questions indicate the amount of work that might be involved in this task.

Question 9.9.*

(a) With 51 voters in the Electoral College (the 50 states plus the District of Columbia), how many different coalitions are possible? Clearly explain your answer. (For this question, you may assume that Maine and Nebraska go along with the winner-take-all rule.)

(b) How long would it take you to form all of the different possible coalitions from part (a) using a computer that could form one million such coalitions per second? Express your answer in years.

Question 9.10.

(a) How many different ways are there to arrange in some order the 51 voters in the Electoral College?

(b) How long would it take you to form all of the different arrangements from part (a) using a computer that could form one million such arrangements per second? Express your answer in years using scientific notation.

As suggested by Questions 9.9 and 9.10, the calculations needed to find the power indices for the states in the Electoral College would require sophisticated mathematical methods and a lot of computing power. While

a discussion of how these power indices could be calculated efficiently is beyond the scope of this book, we can take comfort in knowing that the calculations have been done, and their results are readily available.

The Banzhaf indices of all of the states in the Electoral College for the 2016 presidential election are shown in Table 9.5. Note that the percentage of the total Banzhaf power held by each state differs only very slightly from the percentage of the electoral votes controlled by the state, with the only exception being California. The Shapley-Shubik power percentages for the Electoral College (not shown) are similar to the Banzhaf percentages; they indicate a slightly smaller percentage of power for California, but one that is still noticeably higher than the percentage of the electoral votes controlled by the state. If the population of California continues to grow at its current rate, these gaps will continue to increase.

With that said, it's interesting to note that even with the massive number of electoral votes controlled by California and its high percentage of power in the Electoral College, the state received very little attention from the candidates in the campaign leading up to the 2016 election. This was because early polls indicated that California, which has become a reliably Democratic state, would almost certainly be won by Hillary Clinton. So despite everything we have said about the power held by individual voters in California and the state as a whole, it actually had only a minimal impact on the outcome of the 2016 presidential election. Donald Trump basically conceded the state to Clinton early on, just as Clinton conceded Texas to Trump early on (although Clinton did run some ads in Texas in the late stages of her campaign). Both candidates spent the bulk of their time and money in states that were expected to be close, such as Michigan, Pennsylvania, and Wisconsin, which indeed were each won (by Trump) by a margin of less than 0.8% of the popular votes that were cast in the state.

It's also interesting to note that, even in spite of the apparent advantage enjoyed by larger states in the Electoral College, the winner-take-all rule really does offer some protection for smaller states. For example, consider the final numbers of pledged electoral votes in the 2000 presidential election, which are shown in Table 9.6. Given these numbers, we can see that if Bush had lost Wyoming to Gore in the 2000 presidential election, then Gore would have won the presidency instead of Bush (which, by the way, would have also alleviated the controversy in Florida). So the Electoral College does incentivize campaigning in smaller states, especially in close elections.

States	Electoral Votes	Percentage of Electoral Votes	Percentage of Banzhaf Power
CA	55	10.2%	11.4%
TX	38	7.1%	7.2%
FL, NY	29	5.4%	5.4%
IL, PA	20	3.7%	3.7%
OH	18	3.3%	3.3%
GA, MI	16	3.0%	2.9%
NC	15	2.8%	2.7%
NJ	14	2.6%	2.6%
VA	13	2.4%	2.4%
WA	12	2.2%	2.2%
AZ, IN, MA, TN	11	2.0%	2.0%
MD, MN, MO, WI	10	1.9%	1.8%
AL, CO, SC	9	1.7%	1.6%
KY, LA	8	1.5%	1.5%
CT, OK, OR	7	1.3%	1.3%
AR, IA, KS, MS, NV, UT	6	1.1%	1.1%
NE, NM, WV	5	0.9%	0.9%
HI, ID, ME, NH, RI	4	0.7%	0.7%
AK, DE, DC, MT, ND, SD, VT, WY	3	0.6%	0.6%

TABLE 9.5. 2016 Electoral College Banzhaf power percentages

Candidate	Electoral Votes (Pledged)
George W. Bush	271
Al Gore	267

TABLE 9.6. 2000 U.S. presidential election

Swing Votes and Perverse Outcomes

So far, we've talked about the 2000 U.S. presidential election and how close it was. We've also hinted at the fact that under the Electoral College, how close an election is depends on much more than just the difference in the total numbers of votes cast for each candidate nationwide. For example, in the 2000 election, if only 538 Harry Browne voters in Florida had switched their votes to Al Gore, then Gore would have won the national election rather than George W. Bush (as you observed in part (a) of Question 9.3). And if it were Bush voters instead of Browne voters switching their votes to Gore, it would only have taken half as many, or 269, to swing the outcome from Bush to Gore. That's 269, or 0.00026%, of the 105,405,100 popular votes that were cast in this election nationwide.

Analyzing the 2016 U.S. presidential election is not quite as simple, since Texas was the only state that could have singlehandedly swung the outcome, and Texas was not particularly competitive. To see how close the 2016 election really was, consider the final numbers of pledged electoral votes, which are shown in Table 9.7, and the total numbers of popular votes cast in each state for both Hillary Clinton and Donald Trump, which are shown in Table 9.8.

Candidate	Electoral Votes (Pledged)
Donald Trump	306
Hillary Clinton	232

TABLE 9.7. 2016 U.S. presidential election

Question 9.11.* In the 2016 presidential election in the state of Texas, what was the total difference in the numbers of popular votes cast for Clinton and for Trump? Counting only votes cast for Clinton and Trump in Texas, what was the percentage difference in the numbers of popular votes cast for these two candidates?

Question 9.12. Which state was the closest between Clinton and Trump in the 2016 presidential election? Does your answer depend on whether you measure the vote difference as an actual number of votes or as a percentage?

Question 9.13.* In the 2016 presidential election, what is the smallest number of voters in Texas who could have changed the outcome of the election by changing their votes from Trump to Clinton?

Question 9.14. In the 2016 presidential election, what is the smallest number of voters nationwide (from any state or states) who could have changed the outcome of the election by changing their votes from Trump to Clinton?

State	Clinton Votes	Trump Votes	State	Clinton Votes	Trump Votes
AL	729,547	1,318,255	MT	177,709	279,240
AK	116,454	163,387	NE	284,494	495,961
AZ	1,161,167	1,252,401	NV	539,260	512,058
AR	380.494	684,872	NH	348,526	345,790
CA	8,753,788	4,483,810	NJ	2,148,278	1,601,933
CO	1,338,870	1,202,484	NM	385,234	319,667
CT	897,572	673,215	NY	4,556,124	2,819,534
DE	235,603	185,127	NC	2,189,316	2,362,631
DC	282,830	12,723	ND	93,758	216,794
FL	4,504,975	4,617,886	OH	2,394,164	2,841,005
GA	1,877,963	2,089,104	OK	420,375	949,136
HI	266,891	128,847	OR	1,002,106	782,403
ID	189,765	409,055	PA	2,926,441	2,970,733
IL	3,090,729	2,146,015	RI	252,525	180,543
IN	1,033,126	1,557,286	SC	855,373	1,155,389
IA	653,669	800,983	SD	117,458	227,721
KS	427,005	671,018	TN	870,695	1,522,925
KY	628,854	1,202,971	TX	3,877,868	4,685,047
LA	780,154	1,178,638	UT	310,676	515,231
ME	357,735	335,593	VT	178,573	95,369
MD	1,677,928	943,169	VA	1,981,473	1,769,443
MA	1,995,196	1,090,893	WA	1,742,718	1,221,747
MI	2,268,839	2,279,543	WV	188,794	489,371
MN	1,367,716	1,322,951	WI	1,382,536	1,405,284
MS	485,131	700,714	WY	55,973	174,419
MO	1,071,068	1,594,511			

TABLE 9.8. 2016 U.S. presidential election popular votes by state

As we saw in the last few questions, the Electoral College can be very sensitive to relatively small changes; that is, the potential exists for a relatively small number of swing voters to have a significant impact on the outcome of an election. But how many popular votes must a candidate receive in order to be elected in the first place?

Question 9.15.* The U.S. Census Bureau's 2016 citizens of voting age population estimates for each of the fifty states and the District of Columbia are shown in Table 9.9. For convenience, suppose for this exercise that all citizens of voting age (and only citizens of voting age) are eligible to vote.

State	Voting-Age Citizens	State	Voting-Age Citizens	State	Voting-Age Citizens
AL	3,653,381	KY	3,329,835	ND	571,119
AK	528,248	LA	3,454,978	OH	8,765,154
AZ	4,710,448	ME	1,056,410	OK	2,807,548
AR	2,185,724	MD	4,239,987	OR	2,956,232
CA	25,002,812	MA	4,924,459	PA	9,752,322
CO	3,896,986	MI	7,436,478	RI	784,997
CT	2,584,884	MN	4,007,159	SC	3,677,799
DE	697,148	MS	2,220,616	SD	634,140
DC	504,242	MO	4,567,771	TN	4,919,574
FL	14,441,877	MT	797,198	TX	17,523,904
GA	7,168,068	NE	1,352,947	UT	1,945,001
HI	1,022,704	NV	1,942,764	VT	494,717
ID	1,168,843	NH	1,035,684	VA	6,062,304
IL	9,017,653	NJ	6,154,126	WA	5,081,800
IN	4,856,797	NM	1,470,045	WV	1,451,557
IA	2,310,467	NY	13,704,991	WI	4,340,567
KS	2,074,102	NC	7,296,335	WY	434,584

TABLE 9.9. 2016 citizens of voting age population estimates by state

(a) Construct a scenario in which Donald Trump could have won the 2016 presidential election while receiving only 11 popular votes nationwide.

(b) In the scenario you came up with for part (a), how many votes could Hillary Clinton have received and still have lost to Trump? In this case, what percentage of the popular votes cast would Trump have received? What percentage of the popular votes cast would Clinton have received? (Note: The sum of all 51 of the population estimates in Table 9.9 is 227,019,486.)

(c) What assumptions did you have to make in order to construct the example you gave for parts (a) and (b)? Were these assumptions reasonable? Why or why not?

Question 9.15 demonstrates that under the Electoral College, it is possible, albeit absurdly improbable, for a candidate to win the presidency while receiving only a ridiculously small number and percentage of the popular votes cast in the election. Of course, to construct such a perverse example, one must make several assumptions that are completely unreasonable for

an actual election. So let's consider what would happen if we attempted to make some of our assumptions a bit more realistic.

One starting point would be to assume that voter turnout is roughly the same from state to state. Although this assumption is not entirely accurate, it does rule out the possibility of strange scenarios in which only one person shows up to vote in certain states and all citizens of voting age show up to vote in other states. (You probably used such a scenario in your answer to Question 9.15.) Incidentally, the ratio of popular votes cast to citizens of voting age in the 2016 U.S. presidential election was 60.2%. (In 2012, this ratio was 58.6%.)

Question 9.16. Consider again the 2016 U.S. presidential election.

(a) Assuming that the voter turnout in each state and the District of Columbia was exactly 60.2% of citizens of voting age, and that each voter cast a vote for either Hillary Clinton or Donald Trump, construct a scenario in which Donald Trump could have won the election while receiving less than 25% of the nationwide popular vote. (Recall that the sum of all 51 of the population estimates in Table 9.9 is 227,019,486.)

(b) In your answer to part (a), which states did you assume Trump would win? What made you choose those states?

(c) If you wanted to construct a winning coalition of states for Trump that contained the smallest possible number of popular votes, should you include more larger states or more smaller states? Give a convincing argument to justify your answer.

(d) What assumptions did you have to make in order to construct your scenario for part (a)? Were all of these assumptions reasonable? Why or why not?

The scenario you constructed for Question 9.16 was perhaps somewhat more realistic than the one you came up with for Question 9.15, but you probably still made some assumptions that were not entirely reasonable. For instance, you may have assumed that in certain states, the entire population voted the same way. You may have also assumed that the vote was split almost exactly evenly in other states.

For the next question, we'll try to work entirely within the context of the real world by making only assumptions that we could reasonably expect to be true for all U.S. presidential elections. Of course, in doing so, we'll be stepping into a bit of a gray area, since we'll have to make some personal judgments about what could and could not reasonably occur in an actual election. Some of this is a matter of opinion, and so our task will be twofold. First, we'll have to come up with good arguments to make the case that our assumptions are reasonable. Then, we'll have to use good reasoning to come up with a defensible, worst-case scenario based on these assumptions.

Question 9.17. Complete the following statement:

> *In an actual U.S. presidential election with only two candidates, it would be virtually impossible for a candidate to win the election without receiving at least ____ % of the popular votes.*

Give a convincing argument to justify your answer, describing in detail the assumptions you made, why these assumptions are reasonable, and why your answer follows from them.

Alternatives to the Electoral College

We'll conclude this chapter by briefly considering a few of the alternative methods that have been proposed to replace the Electoral College system. The list below is by no means exhaustive, but it does give a sense of some of the recent debate surrounding the Electoral College. Also note that some of the alternatives listed would be easier to implement than others. For example, any proposal to completely abolish the Electoral College would require an amendment to the U.S. Constitution; proposals that would only involve changing the way states allocate their electoral votes would not require such an amendment.

Alternative Method 1: **Plurality.** The plurality winner of the nationwide popular vote could just be declared the winner. (This would have made Hillary Clinton supporters very happy in 2016.)

Alternative Method 2: **District System.** The method currently used to award electoral votes in Maine and Nebraska could be used nationwide. Under this method, the winner-take-all rule would hold in the District of Columbia and states that have only a single congressional district, but larger states would be able to split their electoral votes among the different candidates.

Alternative Method 3: **Proportional System.** States could choose to allocate their electoral votes proportionally, so that, for example, a candidate who received 27.32% of the popular votes cast in a state would receive 27.32% of the state's electoral votes. This would necessitate the use of fractional electoral votes—or, alternatively, states could choose to round the number of electoral votes awarded to the nearest whole number. (We'll see in Chapter 11 that this latter option is not quite as straightforward as it may seem.) Legislators in several states have proposed bills to implement a proportional system, but most of these bills have failed to gain significant traction.

Alternative Method 4: **Approval Voting.** Approval voting could be used, either as an outright replacement for the Electoral College, or as an alternative to plurality in deciding the winner of each state. Approval voting

would be a relatively simple way to allow voters to better express their preferences in races that involve more than two candidates.

Question 9.18. If the proportional system (Alternative Method 3) had been used in the 2000 U.S. presidential election, how many electoral votes would have been awarded to George W. Bush in Florida?

Question 9.19. Could a candidate win a U.S. presidential election under Alternative Method 1 but lose the same election under Alternative Method 3? What about vice versa? Give a convincing argument to justify each of your answers.

Question 9.20. Briefly discuss any positive or negative features that you can identify in each of the four alternative methods described above. Which of these methods do you think is the best alternative to the Electoral College? Do you think the Electoral College should be replaced by one of these methods? Why or why not?

Question 9.21. Find or invent another alternative method to the Electoral College, and discuss any positive or negative features of this method. Do you think the Electoral College should be replaced by your method? Why or why not?

Question 9.22. Based on what you have learned in this chapter, give an honest assessment of the Electoral College from your own point of view. Do you think it's a good system? Should we continue to use it? Why or why not? If you were a presidential candidate, which would you prefer to be used to declare the winner of your election: plurality or the Electoral College? How would you campaign differently if plurality were used instead of the Electoral College?

Questions for Further Study

Question 9.23. Decide which of the criteria we discussed for evaluating voting systems in Chapters 2–5 can be applied to the Electoral College. Then decide, if possible, which of the applicable criteria the Electoral College satisfies and which it violates. Explain your answers in detail, and give convincing arguments to justify your claims.

Question 9.24. Describe a scenario (using actual state names and electoral vote totals) in which a single faithless elector could change the outcome of a U.S. presidential election.

Question 9.25. The final electoral vote tally from the 2016 U.S. presidential election was 304 votes for Donald Trump and 227 for Hillary Clinton, with 7 faithless electors (2 of which were pledged to Trump and 5 to Clinton) who voted for other people.

(a) Find and write a description of who the faithless electors voted for.

(b) Discuss whether you think it is ever acceptable for an elector to vote in the Electoral College for someone other than the candidate to whom they are pledged.

(c) In Trump's winning coalition of states (the collection of states that he won), figure out which of the states were critical according to how we defined this term in Definition 8.2.

(d) If Texas had been won by Clinton instead of Trump, and if the faithless electors pledged to Clinton had voted for Clinton in the Electoral College, which states would have been critical in Clinton's winning coalition of states?

Question 9.26. The final electoral vote tally from the 2004 U.S. presidential election was 286 votes for George W. Bush and 251 for John Kerry, with 1 faithless elector pledged to Kerry who voted for someone else.

(a) Find and write a description of why the faithless elector voted for someone else.

(b) In Bush's winning coalition of states (the collection of states that he won), figure out which of the states were critical according to how we defined this term in Definition 8.2.

(c) In this election, by how many votes did Bush win the state of Ohio? How many Bush voters would have needed to switch their votes from Bush to Kerry in order to change the outcome of the election in Ohio?

(d) Suppose that exactly the number of voters you specified in part (c) had switched their votes from Bush to Kerry. In this scenario, who would have won the national election?

(e) In the situation described in part (d), what percentage of the nationwide popular vote would Bush have received? What percentage would Kerry have received?

(f) Does anything about your answers to parts (d) and (e) strike you as being strange or unusual? (Hint: Think back to 1876.)

Question 9.27. The final electoral vote tally from the 2000 U.S. presidential election was 271 votes for George W. Bush and 266 for Al Gore, with 1 faithless elector pledged to Gore who abstained from voting.

(a) Find and write a description of why the faithless elector abstained from voting. Then discuss whether you agree or disagree with the elector's point of protest.

(b) Discuss whether you think it is ever acceptable for an elector to abstain from voting in the Electoral College.

(c) In Bush's winning coalition of states (the collection of states that he won), figure out which of the states were critical according to how we defined this term in Definition 8.2.

(d) If Florida had been won by Gore instead of Bush, and if the faithless elector had voted for Gore in the Electoral College, which states would have been critical in Gore's winning coalition of states?

Question 9.28. Research the last three times prior to 2000 that an elector was faithless. Write a detailed summary of your findings.

Question 9.29. Find and write a summary of each U.S. presidential election in which the Electoral College winner did not receive a plurality of the nationwide popular vote.

Question 9.30. Has there ever been a U.S. presidential election with a unanimous Electoral College winner? If so, what percentage of the nationwide popular vote did the winning candidate receive? If not, which U.S. president came the closest to being a unanimous Electoral College winner?

Question 9.31. In your opinion, which of the following U.S. presidential elections was closest: 2000, 1960, or 1880? Explain why you think so, and then give a reason why it could also be argued that each of the other two was actually closer.

Question 9.32. Write a complete summary of how the U.S. presidential election of 1800 concluded despite the fact that there was no majority winner in the Electoral College. Include in your summary a description of how Aaron Burr later exacted his revenge on Alexander Hamilton.

Question 9.33. Write a complete summary of how the U.S. presidential election of 1824 concluded despite the fact that there was no majority winner in the Electoral College. Include in your summary a description of the "corrupt bargain."

Question 9.34.

(a) In a presidential election with exactly two candidates, would it be possible for a candidate to win unanimously in the Electoral College without receiving a plurality of the nationwide popular vote? Give a convincing argument or example to justify your answer.

(b) In a presidential election with more than two candidates, would it be possible for a candidate to win unanimously in the Electoral College without receiving a plurality of the nationwide popular vote? Give a convincing argument or example to justify your answer.

Question 9.35. Find a copy of Article 2, Section 1 of the U.S. Constitution, and write a summary of what it states. Then critique this section of the Constitution by identifying any deficiencies present in its description of how the Electoral College should operate.

Question 9.36. Find a copy of the 12th Amendment to the U.S. Constitution, and write a summary of what it states. Then append to your summary a complete description of the historical event that caused the amendment to be written and ratified in the first place.

Question 9.37. Suppose that in the 2016 U.S. presidential election, every voter in your state voted for either Hillary Clinton or Donald Trump. If, upon the initial counting of the votes, it was discovered that one candidate had defeated the other by the smallest margin possible (either one or two votes, depending on whether there were an even or odd number of voters), would a recount of the votes cast in your state have been required? If so, find out the procedures that would have governed the recount, and write a summary of your findings. Include in your summary exactly how close the initial vote counts would have needed to be for a recount to be necessary.

Question 9.38. Find an article in a popular media source that expresses a positive view of the Electoral College. Write a summary and critique of the article based on what you learned in this chapter.

Question 9.39. Find an article in a popular media source that expresses a negative view or questions the constitutionality of the Electoral College. Write a summary and critique of the article based on what you learned in this chapter.

Question 9.40. Find a statement of the Median Voter Theorem, and write a summary of what it says. Does the Median Voter Theorem accurately reflect the way candidates campaign in U.S. presidential elections? Give a convincing argument to justify your answer.

Question 9.41. In the 2000 U.S. presidential election, some Nader supporters—who thought that their votes for Nader might hurt Gore's overall chances in the election—considered swapping their votes with the votes of Gore supporters from other states. More specifically, Nader supporters in close states (states in which a vote for Nader could have swung the outcome of the election) proposed voting instead for Gore, as long as a Gore voter in a less competitive state cast their ballot for Nader. This would have left the popular vote totals for the two candidates essentially unchanged, thereby giving Nader the publicity he had earned while also increasing Gore's chances of winning in the close states he might have otherwise lost. Do you think this is a sensible way of dealing with some of the problems created by the Electoral College? Why or why not?

Question 9.42. Based on the Banzhaf power percentages in Table 9.5, one could argue that there is a bias in the Electoral College toward larger states. In light of such an argument, how would you explain the fact that some of the strongest advocates of the Electoral College come from smaller states?

Question 9.43. Research the National Pupular Vote Interstate Compact, and write a summary of your findings. Do you think that the compact is a viable mechanism for reforming the Electoral College? Why or why not?

Answers to Starred Questions

9.2. Johnson's win in Texas *might* have changed the outcome. Since no candidate would have received a majority of the 538 electoral votes, the Electoral College would have resulted in a tie, with no winner determined.

9.3. (a) If Gore had received 538 additional votes in Florida, then he would have beaten Bush in Florida by a single vote. Thus, only 538 Browne voters would have needed to switch their votes to Gore in order to change the outcome of the election.

9.5. The electoral votes held by California increased by $\frac{1}{54} = 1.85\%$.

9.7. Each elector represented $\frac{585{,}501}{3} = 195{,}167$ people.

9.9. (a) There are a total of $2^{51} = 2{,}251{,}799{,}813{,}685{,}248$ different coalitions. (Now explain why this is true.)

9.11. The total difference was $4{,}685{,}047 - 3{,}877{,}868 = 807{,}179$, and the percentage difference was $\frac{807{,}179}{4{,}685{,}047+3{,}877{,}868} = 9.43\%$.

9.13. If 403,590 of the Trump voters in Texas had changed their votes to Clinton, then Clinton would have received 4,281,458 votes in Texas, while Trump would have received only 4,281,457.

9.15. Suppose that only one person voted in each of the 11 largest states, while all citizens of voting age voted in each of the other states. Under these circumstances, Trump could have won with only 0.00001092% of the popular votes.

Chapter 10

Trouble in Direct Democracy

Focus Questions

In this chapter, we'll explore the following questions:

- What is a referendum election?
- In a referendum election, what does it mean for a voter's preferences to be separable? What problems can occur when voters' preferences are not separable?
- How can binary preference matrices be used to represent the preferences of voters in referendum elections?
- What options have been proposed for resolving the separability problem? What are the pros and cons of each option?

Warmup 10.1. The administration at Little Valley College (LVC) is facing a serious crisis. Due to an influx of new commuter students, there are now more cars on campus than parking spaces. In an attempt to solve this problem, two proposed solutions are being considered:

Proposal 1. Double the price of a student parking permit (thereby encouraging students to carpool or take the bus).

Proposal 2. Build a new parking garage.

An election will be held to allow LVC's students to decide if either or both of these proposals should be approved. The rules for the election are as follows:

- Voting on the two proposals will be conducted simultaneously; that is, both proposals will appear on the same ballot.
- Each voter must register a vote of *yes* or *no* on each proposal.

- A proposal will pass (i.e., be approved) if and only if a majority of the voters vote *yes* on it.

Dave, Mike, and Pete are three roommates at LVC, and their preferences for the outcome of the election are shown in Table 10.1. In this table, Y denotes passage of a proposal and N denotes failure.

	Outcome of Proposal 1/2		
Rank	Dave	Mike	Pete
1	Y/N	N/Y	N/N
2	N/Y	Y/Y	N/Y
3	Y/Y	Y/N	Y/N
4	N/N	N/N	Y/Y

TABLE 10.1. Preferences for the LVC parking election

(a) Give a reasonable explanation for each of Dave's, Mike's, and Pete's preferences. That is, try to explain intuitively what views or beliefs might have motivated each of their preferences.

(b) If Dave, Mike, and Pete were the only students who showed up to vote, what would the outcome[1] of the LVC parking election be?

(c) Do you think the result from part (b) is a good outcome? Does it accurately reflect the will of the voters? Why or why not?

The election described in Warmup 10.1 is an example of what is commonly known as a *referendum election*. Referendum elections have become increasingly popular in many countries; in the United States, they are used primarily as a way to give voters a direct voice in certain state and local issues. Because referendum elections bypass the representative bodies that typically decide these issues (for instance, state legislatures and city commissions), they are often hailed by proponents as an effective and efficient way to implement *direct democracy*. In fact, according to political scientists Dean Lacy and Emerson Niou [**33**], "the resurrection of direct democracy through referendums is one of the clear trends of democratic politics."

There is definitely a certain appeal to the idea of direct democracy and its implementation through referendum elections. Many have embraced the argument put forth by economist Brian Beedham [**7**] that "direct democracy ... leaves no ambiguity about the answer to the question: What did the people want?" But perhaps we should not be so quick to jump to this conclusion. After all, as we saw in the LVC parking election, referendum elections do not always result in outcomes that truly represent the will of

[1] By the *outcome* of a referendum election we mean the overall result of voting on all of the proposals. So, for instance, the outcome of an election with three proposals might be $Y/Y/Y$, meaning that all three proposals passed. Or the outcome might be $N/Y/N$, meaning that only the second proposal passed.

the voters. In fact, in that example, the outcome of the election was the least preferred choice of two thirds of the voters!

This isn't the first time we've seen an election lead to a strange or paradoxical outcome, but it is the first example we've seen in the context of referendum elections. So it makes sense to ask: What is it about referendum elections that allows this undesirable behavior to occur? And what can we do to address these problems? In this chapter, we'll consider these and other interesting questions. By doing so, we'll learn about the surprising complexities of referendum elections—and thus be able to more carefully evaluate the claims made by both advocates and opponents of direct democracy.

Even More Trouble

The example we looked at in Warmup 10.1 was interesting, but not as interesting as it could have been. In fact, if we modify the situation slightly, we can see that the outcome could have actually been much worse.

Question 10.2. Consider again the LVC parking election from Warmup 10.1.

(a) Suppose Dave and Mike each recruit 10 of their friends to vote the same way they do. Assuming Pete's preferences remain as they were, would the addition of these 20 extra voters change the outcome of the election?

(b) Suppose Dave and Mike each recruit 100 of their friends to vote the same way they do. Again assuming Pete's preferences remain as they were, would the addition of these 200 extra voters change the outcome of the election?

(c) Liah, the president of the LVC math club, makes the following claim:

> *Even if all 25,461 students at Little Valley College showed up to vote in the parking election, it would be possible for the outcome to be the least preferred choice of all but one of the voters.*

Is Liah correct? Give a convincing argument or example to justify your answer.

Question 10.3. In an effort to improve their living conditions, Dave, Mike, and Pete have pooled their money and are preparing to make some upgrades to their apartment. Each of them suggests a single purchase: Dave wants a new cool-touch George Foreman grill, Mike wants a foosball table, and Pete wants faster internet service so he can video chat with his girlfriend who is studying abroad in Spain. All three of the roommates like each of these suggestions, but they also all secretly believe that they do not have

enough money between them to pay for all three. To decide what purchases to make, Pete, inspired by his inordinately powerful role in the parking election, suggests that the matter be settled by a referendum election. He calls for a simultaneous vote on the following three proposals, with each proposal passing if a majority of the voters vote *yes* on it:

Proposal D. Purchase a cool-touch George Foreman grill.

Proposal M. Purchase a foosball table.

Proposal P. Purchase faster internet service.

After the ballots are cast, the three roommates ask their friend Eric to tabulate the results. They wait anxiously until Eric finally returns and announces the result: All three proposals passed!

(a) Give examples of preferences for Dave, Mike, and Pete that would yield this outcome.

(b) Explain how it is conceivable that the outcome of the election could be the least preferred choice of all three of the voters.

(c) Hearing the three roommates' plight, Liah makes another bizarre claim:

> *In a referendum election with an arbitrarily large number of voters, it would be possible for the outcome to be the least preferred choice of all of the voters.*

Is Liah correct this time? Give a convincing argument or example to justify your answer.

(d) What do you think is causing the strange outcomes we've seen in this question and Question 10.2? Explain.

The Separability Problem

As we've seen in the last few questions, referendum elections can sometimes produce outcomes that fail to accurately reflect the preferences of the voters. But why? What causes this undesirable behavior? The next question will help us identify one possible explanation.

Question 10.4.* Consider again the LVC parking election from Warm-up 10.1.

(a) Suppose Dave somehow found out that Mike and Pete were going to vote N/Y and N/N, respectively. Do you think this information might change the way Dave would vote? Why or why not?

(b) Suppose you told Dave that you knew whether Proposal 2 was going to pass or fail, and then asked him whether he wanted Proposal 1 to pass or fail. What do you think he would say? Explain.

(c) Suppose you told Pete that you knew whether Proposal 2 was going to pass or fail, and then asked him whether he wanted Proposal 1 to pass or fail. What do you think he would say? Explain.

(d) In parts (b) and (c), you should have identified a difference between Dave's and Pete's preferences. In light of this difference, are Mike's preferences more like Dave's or Pete's? Clearly explain your answer.

Question 10.4 reveals some important features of voter preferences in referendum elections. These features are central to what some economists and political scientists have called the *separability problem*, which can be summarized as follows:

- In a referendum election, the outcome that a voter wants on one or more proposals could depend on the outcome of other proposals. (For instance, a voter might want Proposal A to pass, but only if Proposal B also passes.)
- Simultaneous voting (voting on all of the proposals at the same time) doesn't allow voters a way to express these kinds of complex preferences. Instead, voters are forced to separate issues that may be linked in their minds.
- Since voters cannot fully express their true preferences, the outcome of the election may not be a good representation of what the voters really want.

In order to more fully understand the separability problem and thus be able to work toward a satisfying solution, we must first understand what it means for a voter's preferences in a referendum election to be separable. The following definition formalizes this idea.

Definition 10.5. Let v be a voter in a referendum election.

- A collection S of one or more proposals in the election is **separable with respect to v** if v's ranking of the possible combinations of outcomes for the proposals in S does not depend on the outcome of any of the proposals not in S.
- The preferences of v are **separable** (or *completely separable*) if every possible collection of one or more proposals is separable with respect to v.

Question 10.6.* Consider again Dave's, Mike's, and Pete's preferences and the proposals in the LVC parking election from Warmup 10.1.

(a) Is Proposal 1 separable with respect to Dave, Mike, or Pete? If so, with respect to which voter(s) is it separable? Explain how you know.

(b) Is Proposal 2 separable with respect to Dave, Mike, or Pete? If so, with respect to which voter(s) is it separable? Explain how you know.

(c) Are any of Dave's, Mike's, or Pete's preferences completely separable? If so, whose? Explain how you know.

Question 10.7. In the example from Question 10.3, are Dave's, Mike's, and Pete's preferences likely to be separable? Why or why not?

Question 10.8. Suppose a voter in a referendum election with three proposals ranks the possible outcomes as follows:

$$Y/Y/Y \succ Y/N/Y \succ Y/N/N \succ Y/Y/N \succ N/N/N \succ N/Y/Y \succ N/Y/N \succ N/N/Y$$

Are this voter's preferences separable? Why or why not?

As you probably observed in Question 10.8, it can be difficult to determine if a voter's preferences in a referendum election are separable. This is primarily due to the fact that the definition of separability requires that we consider every possible collection of proposals. For elections with just two proposals, this task only involves looking at each individual proposal. But for elections with more than two proposals, the situation is not quite so simple.

Question 10.9.* Suppose you wanted to know if a particular voter's preferences in a referendum election were separable.

(a) If there were three proposals in the election, what is the maximum number of collections of proposals you would need to consider?

(b) If there were five proposals in the election, what is the maximum number of collections of proposals you would need to consider?

(c) If there were ten proposals in the election, what is the maximum number of collections of proposals you would need to consider?

(d) In each of parts (a)–(c), why do you think you were asked about the *maximum* number of collections of proposals you would need to consider? Would you ever be able to get by with looking at fewer than this maximum number? Clearly explain your answers.

As we saw in Question 10.9, it can be a lot of work to check if a voter's preferences in a referendum election are separable—especially if the election involves a large number of proposals. Fortunately, however, there are some shortcuts, which we'll learn about soon. But first, let's take a few minutes to explore a mathematical model that provides a convenient way to represent voter preferences in referendum elections.

Binary Preference Matrices

For each voter in a referendum election, we can represent the voter's preferences with a rectangular array of zeros and ones called a **binary preference matrix**. The next question illustrates the correspondence between voter preferences and binary preference matrices.

Question 10.10.* Consider again the LVC parking election from Warm-up 10.1. For this election, the binary preference matrices that result from Dave's, Mike's, and Pete's preferences are shown in Table 10.2. Given the

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Dave Mike Pete

TABLE 10.2. Binary preference matrices for the LVC parking election

preferences in Table 10.1, how do you think the binary preference matrices in Table 10.2 were formed? (Note: If you answer correctly, you will have discovered the general rule for forming and interpreting binary preference matrices.)

Question 10.11. Which of the following arrays of zeros and ones are binary preference matrices? That is, which could be formed from a voter's preferences in a referendum election with two proposals?

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Question 10.12. Write the binary preference matrix that results from the preferences in Question 10.8. Which collections of proposals in the election would be separable with respect to a voter who has these preferences? Which collections of proposals would not be separable? Does using a binary preference matrix make it easier to identify which collections of proposals are separable and which are not? Clearly explain your answers.

Testing for Separability

Now that we understand what binary preference matrices are and how they are formed, we're ready to investigate a couple of tools that can help us more easily test whether a voter's preferences in a referendum election are separable.

Tool 1: Symmetry

Definition 10.13.

- The **bitwise complement** of a row in a binary preference matrix is formed by interchanging all of the zeros and ones in the row (i.e., by replacing all of the zeros with ones and all of the ones with zeros).
- A binary preference matrix is **symmetric** if for every row number i, the ith row from the top is the bitwise complement of the ith row from the bottom.

Question 10.14.* Which of the binary preference matrices in Table 10.2 are symmetric? Which are not? Explain your answers for each one.

Question 10.15. The top half of a symmetric binary preference matrix is shown below. Find its bottom half.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ & \vdots & \end{pmatrix}$$

If we know that the binary preference matrix associated with a voter's preferences in a referendum election is symmetric, we can conclude quite a bit about the voter's preferences. For instance, if we know what the most preferred outcome is, we can easily determine what the least preferred outcome would have to be. And, as we just saw in Question 10.15, if we know the top half of a symmetric binary preference matrix, we can easily determine what the bottom half would have to be.

As you may have noticed in Question 10.14, the only symmetric binary preference matrix in Table 10.2 was the one associated with a voter whose preferences we had previously determined to be separable. This suggests that perhaps a relationship exists between separable preferences and symmetric binary preference matrices. While it would be a mistake to jump to a conclusion about this relationship based only on the examples from Question 10.14, the following theorem confirms our suspicions.[2]

Theorem 10.16. *If a voter's preferences in a referendum election are separable, then the binary preference matrix associated with these preferences will be symmetric.*

[22]Theorem 10.16 is the first of several results in this chapter that we will state but not prove. This doesn't mean that the proofs are extremely difficult or that you couldn't understand them. They simply involve some notation that we haven't used and a slightly more formal (i.e., less intuitive) approach to the idea of separability.

Question 10.17.* What do Theorem 10.16 and your answer to Question 10.12 allow you to conclude about the preferences in Question 10.8?

Question 10.18.* If the binary preference matrix resulting from a voter's preferences in a referendum election is symmetric, must the voter's preferences be separable? Give a convincing argument or example to justify your answer.

So we can see that while the *non*-symmetry of a binary preference matrix can allow us to conclude that a voter's preferences are *not* separable, the symmetry of a binary preference matrix doesn't help us at all if we are trying to show that a voter's preferences *are* separable. For that, we need another tool.

Tool 2: Unions and Intersections

When examining voters' preferences in a referendum election, we might intuitively expect that the separability of certain collections of proposals would be related to the separability of other collections of proposals. For instance, if we knew that Proposal 1 by itself was separable with respect to a particular voter v, and Proposal 2 by itself was also separable with respect to v, we might expect that Proposals 1 and 2 together would be separable with respect to v as well. Is this in fact the case? To find out, let's look at an example.

Question 10.19.* Consider the following binary preference matrix, which represents a voter's preferences in a referendum election with three proposals.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(a) Are the voter's preferences separable? Why or why not?

(b) Which collections of proposals are separable with respect to the voter? Explain how you know.

(c) Based on your answer to part (b), what can you conclude about the following statement:
In a referendum election, if two collections of proposals S and T are separable with respect to some voter's preferences, then the union of S and T (i.e., the collection

of proposals belonging to either S or T or both) is also separable with respect to the voter's preferences.

The observations you made in Question 10.19 may seem a bit counter-intuitive at first, so let's take a moment to associate the matrix given in that question with a concrete example. For the sake of this example, imagine that each column represents an ingredient that one could include in a dessert: The first column represents chocolate syrup, the second milk, and the third ice cream. Then each row represents a possible dessert, depending on which ingredients are included. For example, mixing all three ingredients together would make a chocolate milkshake, whereas just chocolate syrup and ice cream would make a sundae. If we view the preference matrix from Question 10.19 in this context, we can see that it is entirely reasonable. Notice the following:

- For each individual ingredient, the preferences of the "voter" are separable. For example, for every possible ice cream/milk combination, the voter always prefers having chocolate syrup to not having it.
- If the voter knows that they'll be having ice cream, then their first choice on the chocolate syrup/milk combination is to have both, and their second choice is to just have chocolate syrup.
- If the voter knows that they *won't* be having ice cream, then their first choice on the chocolate syrup/milk combination is still to have both, but their second choice is to just have milk. (Presumably, even though a glass of milk is a boring dessert, it is more tolerable than a glass of straight chocolate syrup.)
- Because the voter's second choice on the chocolate syrup/milk combination depends on whether they are having ice cream or not, the first two ingredients—though individually separable—are not separable when viewed together.

The above example illustrates one way in which a voter's preferences on the various proposals in an election can depend on each other in complex and nuanced ways. In many cases—for example, if we only care about what a voter's *first* choice is on each possible collection of proposals—it suffices to check whether or not each individual proposal is separable. In cases where we want to dig a bit deeper, we can take advantage of the following result that deals not with *unions*, but rather with *intersections*. If you're not already familiar with this term, the intersection of two sets S and T is denoted by $S \cap T$ and is defined to be the set of all elements that belong to *both* S and T.

Theorem 10.20. *If S and T are collections of proposals in a referendum election, and both S and T are separable with respect to a particular voter, then their intersection $S \cap T$ is also separable with respect to the voter.*

Question 10.21.* Suppose that in a referendum election with four proposals, you know that all of the following collections of proposals are separable with respect to some particular voter:

$$\{A, B, C\}, \{A, C, D\}, \{B, C, D\}$$

Which other collections of proposals, if any, would also have to be separable with respect to the voter? Give a convincing argument to justify your answer.

Question 10.22. Suppose you know that in a referendum election with n proposals (where n just represents some arbitrary number), every possible collection of $n - 1$ proposals is separable with respect to some particular voter. Which other collections of proposals, if any, would also have to be separable with respect to the voter? Give a convincing argument to justify your answer.

Some Potential Solutions

Now that we've seen how the notion of separability can affect the outcome of referendum elections, we'll conclude this chapter by exploring some of the different strategies that have been proposed for resolving the separability problem. We'll begin with the most obvious solution.

Potential Solution 1: Avoid Nonseparable Preferences

As we saw in our earlier examples, nonseparable preferences can cause all sorts of undesirable and even paradoxical election outcomes. The following theorem is a natural counterpart to this observation.

Theorem 10.23. *In a referendum election in which every voter has separable preferences, a Condorcet winning outcome will be selected whenever one exists.*

Question 10.24.

(a) Does Theorem 10.23 imply that in a referendum election in which every voter has separable preferences, the winning outcome cannot be the least preferred choice of every voter? Explain.

(b) In a referendum election in which every voter has separable preferences, can the winning outcome be the least preferred choice of every voter? Give a convincing argument or example to justify your answer.

Another result worth noting is the following theorem, which concerns avoiding the possibility of manipulation in referendum elections.

Theorem 10.25. *In a referendum election in which every voter has separable preferences, there will never be a situation in which a voter can guarantee*

a more desirable outcome by voting insincerely (i.e., by voting for an outcome other than their most preferred choice).

Theorems 10.23 and 10.25 both yield strong, positive conclusions. However, they also have very strong hypotheses. To apply either of these theorems, the preferences of *every single voter* in the election must be separable, meaning that even one instance of nonseparability can nullify the theorems' conclusions. This phenomenon is illustrated in the next question.

Question 10.26. Consider again the LVC parking election from Warm-up 10.1, and suppose Dave, Mike, and Pete revise their preferences, resulting in the new binary preference matrices shown in Table 10.3.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Dave Mike Pete

TABLE 10.3. Revised binary preference matrices

(a) Which of Dave's, Mike's, and Pete's revised preferences are separable? Which are not separable? Explain how you know.

(b) According to these revised binary preference matrices, and assuming Dave, Mike, and Pete are the only voters in the election, is there a Condorcet winning outcome? If so, will this outcome be selected as the overall winner?

(c) Construct an example to show that in a referendum election with an arbitrarily large number of voters, all but one having separable preferences, a Condorcet winning outcome can fail to be selected as the overall winner. (Hint: Use the binary preference matrices in Table 10.3.)

Potential Solution 2: Set-wise Voting

Since the separability problem is a result of asking voters to separate issues that may be linked in their minds, another way to solve the problem would be to simply not ask voters to make this separation. That is, instead of viewing a vote of $Y/Y/N$ as separate votes on three proposals (votes of *yes* on the first two proposals and *no* on the third), we could view it as a single vote for the outcome $Y/Y/N$ on all three proposals together. Even better, if we did this we could also allow voters to register their entire preference ballots, and then just use our favorite method from Chapters

2–5 (such as plurality, the Borda count, instant runoff, approval voting, etc.) to decide the winning outcome. This technique is often called *set-wise* voting.

Question 10.27. Consider again the example from Question 10.3, and suppose Dave's, Mike's, and Pete's preferences for the election result in the binary preference matrices shown in Table 10.4.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Dave Mike Pete

TABLE 10.4. Binary preference matrices for the apartment election

(a) Assuming Dave, Mike, and Pete each vote for their most preferred outcome, what outcome would be produced by set-wise voting along with your favorite voting system from Chapters 2–5?

(b) Do you think the outcome under set-wise voting in part (a) more accurately reflects the will of the voters than the outcome under the standard proposal-by-proposal method? Why or why not?

Question 10.28. Write a short discussion of the pros and cons of set-wise voting. Do you think set-wise voting should be used instead of the standard proposal-by-proposal method for all referendum elections? If so, explain why. Otherwise, describe the types of elections for which set-wise voting would be most appropriate.

Potential Solution 3: Sequential Voting

The separability problem ultimately boils down to a lack of information; voters may need information about the outcomes of some of the proposals in a referendum election before they can accurately voice their preferences on other proposals. It would seem natural then to attempt to provide this

information to voters by conducting a sequence of elections instead of requiring voters to vote on all of the proposals at the same time. To see how such a method might work, let's look at an example.

Question 10.29.* Consider again the LVC parking election from Warm-up 10.1, but suppose that instead of voting on both proposals simultaneously, the election is to be held in the following two phases:

- First, the voters will vote on Proposal 1, and the outcome will be announced.
- Then, in a separate election, the voters will vote on Proposal 2.

Assuming Dave, Mike, and Pete are the only voters, and that their preferences are exactly as shown in Table 10.1, would the outcome of the election under this two-phase sequential system be different from the outcome under simultaneous voting? If so, which method (the sequential system or simultaneous voting) do you think results in an outcome that more accurately reflects the will of the voters? Explain.

As we saw in Question 10.29, multi-phase sequential voting can result in outcomes that are better than those resulting from simultaneous voting. But will it always?

Question 10.30. Suppose that in a referendum election with three proposals and three voters, the preferences of the voters are as follows.

- **Voter 1:** $Y/N/Y \succ Y/Y/N \succ Y/Y/Y \succ \cdots \succ Y/N/N$
- **Voter 2:** $Y/Y/N \succ Y/Y/Y \succ Y/N/Y \succ \cdots \succ Y/N/N$
- **Voter 3:** $N/Y/Y \succ Y/N/N \succ Y/Y/N \succ Y/Y/Y \succ \cdots \succ Y/N/Y$

(a) What would the outcome of the election be under simultaneous voting?

(b) Suppose the election is to be held sequentially in two phases, with the voters first voting on Proposal 1 and then, after its outcome is announced, on Proposals 2 and 3 simultaneously. What would the overall outcome of the election be under this two-phase sequential system? Do you think this outcome is better or worse than the outcome from part (a)?

(c) Suppose the election is to be held sequentially in three phases, with the voters first voting on Proposal 1, then on Proposal 2, and then on Proposal 3. What would the overall outcome of the election be under this three-phase sequential system? Do you think this outcome is better or worse than the outcome from part (a)?

(d) Suppose the election is to be held sequentially in two phases, with the voters first voting on Proposals 1 and 2 simultaneously, and

then on Proposal 3. What would the overall outcome of the election be under this two-phase sequential system? Do you think this outcome is better or worse than the outcome from part (a)?

Question 10.30 shows (surprisingly!) that sequential voting does not always result in an outcome that is better than the outcome of simultaneous voting. In fact, the outcome under simultaneous voting can actually be more desirable to a majority of voters than the outcome for the same election under sequential voting. As it turns out, the additional information provided by sequential voting can at times be detrimental rather than beneficial. In addition, the outcome of a sequential election can depend on the order in which the voting takes place (much like the outcome of sequential pairwise voting can depend on the agenda), which introduces the potential for manipulation. Finally, sequential voting can be costly and time-consuming, particularly for elections with a large number of proposals.

With all of that said, sequential voting can be helpful (or at least not harmful) in one very special case.

Theorem 10.31. Suppose that in a referendum election, the voters first vote simultaneously on all but one of the proposals (with the outcomes announced), and then on the remaining proposal. Let O be the outcome of the election under this two-phase sequential system, and assume that every voter votes sincerely (i.e., for their most preferred outcome). Then all of the following statements will be true:

- It is impossible for the outcome under simultaneous voting to be preferred to O by a majority of voters.
- It is impossible for O to be a Condorcet losing outcome.
- It is impossible for O to be the least preferred choice of every voter.

Unfortunately, the scenario described in Theorem 10.31 is the only one in which sequential voting has been shown to be a consistently effective solution to the separability problem. And even in that very special case, there are still some problems that must be resolved.

Question 10.32. If you wanted to use the method described in Theorem 10.31 to decide the outcome of a referendum election, how would you decide which proposal should be voted on last? Clearly explain your answer.

Question 10.33. In a referendum election with just two proposals, could the outcome produced by simultaneous voting ever be preferred by a majority of voters to an outcome produced by sequential voting? Why or why not?

Potential Solution 4: Contingent Ballots

Question 10.34. Consider again the LVC parking election from Warm-up 10.1, but now suppose that each voter is provided with a ballot containing

the following questions, each of which must be answered with a vote of *yes* or *no*:

- Should Proposal 1 be approved?
- Assuming Proposal 1 is approved, should Proposal 2 be approved?
- Assuming Proposal 1 is not approved, should Proposal 2 be approved?

(a) Explain how such a ballot (often called a *contingent ballot*) could be used to mimic sequential voting in a referendum election.

(b) What are some of the advantages and disadvantages of using contingent ballots instead of sequential voting?

(c) What are some of the advantages and disadvantages of using contingent ballots instead of simultaneous voting?

(d) In a referendum election with two proposals, could the outcome produced by simultaneous voting ever be preferred by a majority of voters to an outcome resulting from contingent ballots? Why or why not?

Potential Solution 5: Iterative Voting

One recent proposal to solve the separability problem involves allowing voters to change their ballots as many times as they want during a fixed voting period (say, one week), with the current results of the vote—based on the ballots that have been cast already—announced in real time. This system, called *iterative voting*, gives voters the opportunity to strategically revise their votes if they see that voting for their most preferred outcome is unlikely to yield a favorable result. The winning outcome is determined by the votes at the end of the voting period, regardless of what votes have been cast previously or how many times voters have changed their ballots. To illustrate how this might work, let's look at a couple of examples.

Question 10.35.* Consider again the LVC parking election from Warm-up 10.1, and suppose that Dave, Mike, and Pete agree to use iterative voting to decide the outcome. Initially, suppose each of the roommates votes for their most preferred outcome.

(a) A little while after Dave casts his (first) ballot, he checks the results and sees that Proposals 1 and 2 are both failing, with each receiving one *yes* vote and two *no* votes. Explain how Dave could change his vote to ensure a better outcome for himself.

(b) If Dave made the change you identified in part (a), would either Mike or Pete have an incentive to make subsequent changes to their votes?

(c) What do you think the final outcome of the iterative voting election would be? Is this outcome better or worse than the outcome of simultaneous voting? Explain.

Question 10.36. If the election from Question 10.3 was conducted using iterative voting, what do you think the outcome would be? Clearly explain your reasoning, including which voters might change their ballots and how many changes they might make. (You may want to assume that the voters' preferences are as given in Question 10.27.)

As of this writing, the idea of iterative voting remains purely theoretical; there have been no documented cases of iterative voting actually being used to decide the outcome of a referendum election. However, computer simulations have shown that iterative voting produces results that are often better—and very rarely worse—than those of simultaneous voting.

Question 10.37. What do you think are the advantages and disadvantages of iterative voting? What practical considerations would need to be addressed in order to conduct an election using iterative voting?

Potential Solution 6: To Be Determined

Question 10.38. There is still a lot to learn about the separability problem, and much of the recent research has involved undergraduate students. With that inspiration, suggest a potential solution to the separability problem that is different from those discussed in this chapter. Analyze the pros and cons of your potential solution, and describe the types of situations for which it would be best suited.

Questions for Further Study

Question 10.39. In a referendum election with two proposals and any number of voters, is it possible for the outcome to be the least preferred choice of every voter? Give a convincing argument or example to justify your answer.

Question 10.40.

(a) In a referendum election in which every voter has separable preferences and votes sincerely, is it possible for a Condorcet losing outcome to be selected by simultaneous voting? Give a convincing argument or example to justify your answer.

(b) Repeat part (a), but this time assume that all but one of the voters have separable preferences.

Question 10.41. Suppose that in a referendum election with three proposals, a particular voter's preferences can be described as follows:

- The voter's most preferred outcome is for all three proposals to pass, and the voter's least preferred outcome is for all three to fail.

- The voter prefers any outcome in which two proposals pass over any outcome in which only one proposal passes.

Could this voter's preferences be separable? Could they be nonseparable? Give a convincing argument or example to justify each of your answers.

Question 10.42.

(a) Make a list of every possible binary preference matrix for a referendum election with two proposals.

(b) Which of the binary preference matrices you listed in part (a) are symmetric?

(c) Which of the binary preference matrices you listed in part (a) result from preferences that are separable?

(d) Based on your answers to parts (a)–(c), can you characterize the relationship between separable preferences and symmetric binary preference matrices in referendum elections with two proposals?

Question 10.43.

(a) In a referendum election with two proposals, how many different binary preference matrices are possible? How many are symmetric?

(b) In a referendum election with three proposals, how many different binary preference matrices are possible? How many are symmetric?

(c) In a referendum election with n proposals (where n just represents some arbitrary number), how many different binary preference matrices are possible? How many are symmetric?

(d) Using Theorem 10.16 and your answers to parts (a)–(c), explain why the likelihood of a randomly selected voter in a referendum election having separable preferences will decrease toward zero as the number of proposals increases.

(e) In 1990, a referendum election was held in California that, in addition to a number of local initiatives, contained 28 statewide proposals. How likely do you think it was that any of the voters in the election had nonseparable preferences? Explain.

Question 10.44. Find out the details of the Turkish constitutional referendum election in April 2017, and write a summary of your findings. Include in your summary some information about the reasons for and history of the referendum, and its outcome and aftermath.

Question 10.45. Find out the details of the Maine Ranked Choice Voting Initiative from November 2016, and write a summary of your findings. Include in your summary some information about the reasons for and history of the initiative, and its outcome and aftermath.

Question 10.46. Find out the details of California's Drug Price Standards Initiative from November 2016, and write a summary of your findings. Include in your summary some information about the reasons for and history of the initiative, and its outcome and aftermath.

Question 10.47. Find out the details of the United Kingdom European Union membership referendum election in June 2016, and write a summary of your findings. Include in your summary some information about the reasons for and history of the referendum, and its outcome and aftermath.

Question 10.48. Find out the details of the referendum election on the political status of Puerto Rico in November 2012, and write a summary of your findings. Include in your summary some information about the reasons for and history of the referendum, as well as its outcome and aftermath. Do you think the voters' preferences in the election were separable? Why or why not? How does the outcome of this referendum compare to the outcome of a similar referendum in June 2017?

Question 10.49. In the state of Colorado, a proposal was added to the November 2004 presidential election ballot that could have changed the way the state's electoral votes were allocated in that very same election. Find out the details of this proposal, and write a summary of your findings. Include in your summary a description of the proposed new method for allocating the state's electoral votes, who added the proposal to the ballot, why they did so, and the outcome and aftermath of the referendum. Do you think the voters' preferences on the entire ballot were likely to have been separable? Why or why not?

Question 10.50. Find out the details of a recent referendum election in your state, and write a summary of your findings. Include in your summary a statement of each of the proposals in the election and the outcome. Do you think that some of the voters in the election might have had nonseparable preferences? If so, describe any potential links between the proposals in the election, and explain how these links could have affected the separability of the voters' preferences.

Question 10.51. Suppose a voter's preferences correspond to a symmetric binary preference matrix. Could this voter's preferences be completely nonseparable? In other words, is it possible for every collection of proposals to be nonseparable with respect to such a voter's preferences? Give a convincing argument or example to justify your answer.

Question 10.52. For each of the parts below, find a preference matrix corresponding to a voter in an election with three proposals whose preferences are separable on exactly the sets listed (and no others). If you don't think it is possible to find such a matrix, explain why.

(a) $\{1\}$, $\{2\}$

(b) $\{1, 2\}$, $\{3\}$

(c) $\{1\}, \{2\}, \{3\}, \{1,2\}$

(d) $\{1\}, \{1,2\}, \{2,3\}, \{3\}$

Question 10.53. Suppose that the first (or last) two rows of a binary preference matrix are bitwise complements of each other. What can you conclude about the corresponding preferences, and why?

Question 10.54. Call a voter's preferences *monoseparable* if they are separable on each individual proposal, but not necessarily on larger sets of proposals. If a voter has monoseparable preferences, must their corresponding binary preference matrix be symmetric? Why or why not?

Question 10.55. Design a contingent ballot that could be used in a referendum election with three proposals. How many questions would such a ballot need to contain?

Answers to Starred Questions

10.4. (a) This information should change the way Dave would vote. If he voted for his first choice, Y/N, then his last choice, N/N, would be the winning outcome. But if he voted (insincerely) for N/Y (his second choice), then N/Y would be the winning outcome.

(b) He would say that it would depend on whether Proposal 2 was going to pass or fail.

(c) He would say that he would want Proposal 1 to fail regardless of whether Proposal 2 was going to pass or fail.

10.6. (a) Proposal 1 is separable with respect to Pete's preferences, but not Dave's or Mike's. (Can you explain why?)

(b) Proposal 2 is separable with respect to Mike's and Pete's preferences, but not Dave's.

(c) Pete's preferences are separable, but neither Dave's nor Mike's are.

10.9. (b) With five proposals, the maximum number of collections of proposals you would need to consider is $\binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} = 30$. (See Chapter 8 for a description of this notation.)

10.10. In each binary preference matrix, a 1 represents a Y and a 0 represents an N. Each row represents one possible outcome of the election, and the rows are listed in order of preference, with the voter's most preferred outcome at the top and least preferred outcome at the bottom.

10.14. The binary preference matrix corresponding Pete's preferences is symmetric, but those corresponding to Dave's and Mike's preferences are not.

10.17. The binary preference matrix corresponding to the preferences in Question 10.8 is not symmetric. Therefore, the preferences in Question 10.8 are not separable.

10.18. The voter's preferences do not have to be separable. You may need to look at referendum elections with more than two proposals to find an example of this.

10.19. (a) The voter's preferences are not separable since the binary preference matrix is not symmetric.

(b) Each proposal by itself is separable with respect to the voter. Also, the first and third proposals together are separable with respect to the voter, as are the second and third proposals together. No other collection of proposals is separable with respect to the voter.

(c) For a voter whose preferences correspond to the binary preference matrix in this question, the first two proposals individually are each separable with respect to the voter, but together they are not separable. Thus, the statement is false.

10.21. The following collections of proposals would also have to be separable with respect to the voter: $\{A, C\}$, $\{B, C\}$, $\{C, D\}$, and $\{C\}$.

10.29. The outcome under the sequential system would be N/Y, which is preferred by two of the three voters over the outcome under simultaneous voting.

10.35. (a) Dave could ensure a better result by voting N/Y instead of Y/N. The outcome would then be N/Y, which he prefers to N/N.

(b) Mike would not have an incentive to change his vote, since the result after Dave's change is N/Y—Mike's most preferred outcome. The only outcome Pete prefers to N/Y is N/N, and his vote of N/N already gives him the best chance of obtaining this outcome.

(c) Based on the answers to parts (a) and (b), it seems likely that iterative voting would stabilize on an outcome of N/Y.

Chapter 11

Proportional (Mis)representation

Focus Questions

In this chapter, we'll explore the following questions:

- What method is currently used to apportion the seats in the U.S. House of Representatives? What other methods have been used in the past?
- What is the quota rule? Which apportionment methods satisfy the quota rule, and which methods violate it?
- What are some examples of apportionment paradoxes? Is every apportionment method capable of producing paradoxes?
- Which apportionment method is the best? Is there any one method that is perfect or at least better than the others?

Warmup 11.1. Round off the following fifteen numbers (i.e., turn them into whole numbers) so that the sum of the rounded numbers equals the sum of the unrounded numbers (which is exactly 105):

6.408, 1.594, 2.226, 1.987, 8.622, 12.814, 3.826, 4.965,
9.175, 10.651, 11.693, 1.864, 6.716, 2.301, 20.158.

What rounding method did you use? Describe in detail how you decided which numbers to round up to the next whole number and which to round down to the previous whole number.

So what did you do with the fifteen numbers in Warmup 11.1? The most obvious thing to do would be to round them *conventionally*—that is, to round up if the number's decimal part is bigger than or equal to

.50, and round down otherwise. However, there is a problem with this method: The sum of the rounded numbers equals 106, 1 more than the sum of the unrounded numbers. So you would have to reduce one of the rounded numbers by 1, but which one? Would you reduce the number you rounded up whose decimal part was the smallest? Or maybe the one whose decimal part was the smallest *percentage* of the entire number? (For example, the decimal part of 1.594 is $\frac{.594}{1.594} = 37.26\%$ of the entire number, while the decimal part of 8.622, although larger in size, is only $\frac{.622}{8.622} = 7.21\%$ of the entire number.) Unless you were feeling particularly creative, you probably used one of these two rounding methods—most likely the first.

Question 11.2. Repeat Warmup 11.1, but use one or both of the two rounding methods described in the previous paragraph that you didn't use before.

Which rounding method do you think is better? And why does it matter in the first place? Well, the answers to these questions are more complicated than you might expect, and that is precisely why this chapter and the subject of *apportionment* exist. Apportionment basically deals with the problem of rounding off collections of numbers so that the sum of the rounded numbers equals the sum of the unrounded numbers.[1]

As dry as that may seem, apportionment has a substantial application in the political world, and is in fact at the center of a fascinating piece of American political history. This is because the most important type of apportionment problem is that of allocating seats in a legislative body to a collection of states or districts—and doing so in a way that yields a representation that is proportional to the populations of the states or districts. This problem first arose when the Founding Fathers were trying to decide how to distribute the seats in the U.S. House of Representatives to the original states.

The U.S. House of Representatives

It all started way back on May 25, 1787, when delegates from twelve of the original thirteen colonies met in Philadelphia for the Constitutional Convention. The most intense debate at the convention concerned the organization of the new country's legislature. The larger states wanted representation to be proportional to the states' populations. But, of course, the smaller states wanted all states to have equal representation. The delegates at the convention, in a masterful stroke of ingenuity, came up with a solution that satisfied everyone—both equal and proportional representation, in the Senate and House of Representatives, respectively. This is described in Article 1, Section 1 of the Constitution.

[1] According to Webster's Dictionary, the word *apportion* means "to divide and share out according to a plan; to make a proportionate division or distribution of."

Article 1, Section 2 describes how the House of Representatives should be formed. Clearly, the intent of what is stated there is that the representatives should be allocated to the states based on their populations, but it is never written anywhere in the Constitution exactly how this allocation should be done. The writers of the Constitution surely didn't exclude this important detail without reason. They probably thought it was something that could be decided fairly easily at a more appropriate time, like perhaps a few years later when the initial apportionment of the seats in the House would take place. But, as it turns out, this omission from the Constitution led to much debate, anger, and ongoing frustration among the members of Congress, as well as much research and study by scholars concerning the mathematical problem of apportionment.

In defense of the writers of the Constitution, it's likely that they honestly had no idea just how serious the problems are that result from the noble goal of proportional representation. Apportionment had never really been studied in detail before the U.S. House was formed, and on the surface, the apportionment problem doesn't look too bad. But what happens when we look a bit deeper?

Let's start with the very first apportionment problem Congress ever faced. It occurred in 1794, following and based upon the population figures from the first national census, which took place in 1790. By that time there were fifteen states, and the census of 1790 counted the figures shown in Table 11.1.[2]

Congress needed to allocate exactly 105 seats in the House of Representatives to these fifteen states. After some debate, an apportionment bill authored by Alexander Hamilton was passed by Congress and forwarded to President George Washington.

Hamilton's Apportionment Method

Any apportionment method starts with the idea of a **standard quota**, which is just the exact number of seats a state would be entitled to based on its population. For example, since $\frac{59{,}096}{3{,}893{,}874} = 1.518\%$ of the entire U.S. population lived in Delaware in 1790, Delaware was entitled to exactly $.01518 \times 105 = 1.594$ seats in the first apportioned U.S. House of Representatives. We call 1.594 the standard quota for Delaware, since it is the exact number of seats Delaware would have been entitled to if fractional seats had been possible.

But herein lies the very essence of the apportionment problem: Fractional seats are not possible! So Delaware, while entitled to its standard quota of 1.594 seats, had to be given a whole number of seats. Assuming

[22]The population figures actually used by Congress for the 1794 apportionment were slightly different from the figures shown in Table 11.1, though, since the figures actually used did not fully include the number of slaves and native Americans who lived in the U.S. in 1790.

State	Population
Connecticut	237,655
Delaware	59,096
Georgia	82,548
Kentucky	73,677
Maryland	319,728
Massachusetts	475,199
New Hampshire	141,899
New Jersey	184,139
New York	340,241
North Carolina	395,005
Pennsylvania	433,611
Rhode Island	69,112
South Carolina	249,073
Vermont	85,341
Virginia	747,550
TOTAL	3,893,874

TABLE 11.1. Population totals by state, 1790

this number is either 1 or 2, which number should it have been given? It would seem logical for Delaware to receive two seats, since this is the whole number that 1.594 rounds to conventionally. But conventional rounding, if applied similarly to the standard quotas for the other fourteen states, may not allocate a total of exactly 105 seats, just as conventionally rounding the numbers in Warmup 11.1 does not yield rounded numbers whose sum is 105. In fact, the standard quotas for the fifteen states shown in Table 11.1 are exactly the 15 numbers from Warmup 11.1. So if we tried to do the 1794 apportionment of the seats in the House by rounding the standard quotas for the states conventionally, we would give away 106 seats, one too many. What should we do then instead of conventional rounding so that we give away exactly 105 seats? Well, here's what Alexander Hamilton said to do:

Hamilton Step 1: Find the standard quota for each state.

Hamilton Step 2: Give each state a number of seats equal to its standard quota rounded down.

Hamilton Step 3: See how many seats are left to be allocated (called *surplus seats*), and give those seats one at a time to the states whose standard quotas have the largest decimal parts.

Question 11.3.* Use Hamilton's method to apportion 105 seats to the fifteen states with population figures shown in Table 11.1. Which states were winners under Hamilton's method, and which were losers?

At this point, our little U.S. history lesson takes its first unexpected turn. When President George Washington received a bill from Congress approving Hamilton's method for the 1794 apportionment of the seats in the House, he rejected it and sent it back stamped with the first presidential veto in U.S. history!

There has been considerable speculation as to why Washington vetoed the bill. Some conspiracy theorists claim he did so because his home state, Virginia, was a loser in the method, receiving 20 seats despite a standard quota of 20.158. Others claim he did so at the urging of Thomas Jefferson, who was Washington's Secretary of State, a personal friend, and a fellow Virginian. And indeed, the apportionment method that was eventually adopted was authored by Jefferson himself, and gave Virginia 21 seats. We will choose to believe, however, that Washington's veto was not motivated by self-interest, as Hamilton's method has some serious problems that reveal themselves upon closer inspection.

For one thing, decimal parts of standard quotas aren't always directly comparable. For example, in Question 11.3, Hamilton's method gives a surplus seat to Maryland, whose standard quota has a decimal part of .622, instead of Delaware, whose standard quota has a decimal part of .594—even though Delaware's decimal part is a much larger percentage of its entire standard quota than Maryland's.

Question 11.4.* For the standard quotas for Delaware and Maryland that you determined in Question 11.3, calculate the percentage that each state's decimal part is of its entire standard quota.

Question 11.5.

(a) For the standard quotas for each of the fifteen states in Question 11.3, which state's decimal part makes up the largest percentage of its entire standard quota? Which makes up the smallest percentage?

(b) In light of your answer to part (a), which state do you think was treated best in the apportionment from Question 11.3? Which do you think was treated worst?

In both of Questions 11.3 and 11.5, we looked at the decimal part of each state's standard quota to decide how the state was treated by Hamilton's apportionment method. But, as you might imagine, we could use other criteria as well. For instance, we could consider for each state the average number of residents represented by each of the state's representatives.

Question 11.6.* In the apportionment from Question 11.3, find the average number of residents represented by each of Delaware's representatives. Then repeat this calculation for each of the other states.

Question 11.7. According to your calculations from Question 11.6, which state was treated best in the apportionment from Question 11.3? Which state was treated worst?

So what should we do about poor Delaware? It was the first state, after all. But it's really hard to justify giving Delaware two seats instead of one, as the next question demonstrates.

Question 11.8. Repeat Question 11.6, but this time assume Delaware had two representatives instead of just one. How does Delaware compare to the other states now?

And what about Rhode Island? Does it really deserve to be treated best? Let's consider what would happen to the final figures in Question 11.6 if we took a seat from Rhode Island. Of course, to keep the total number of representatives at 105, we would have to give that seat to some other state. Since Virginia's population is so much larger than any of the other states, its final figure in Question 11.6 would be the least affected if we gave it the extra seat. So let's do this and see what happens.

Question 11.9.* Repeat Question 11.6, but this time assume Rhode Island had just 1 representative instead of 2, and Virginia had 21 representatives instead of just 20. (Note: The calculations from Question 11.6 will be different only for Rhode Island and Virginia.) How do Rhode Island and Virginia compare to the other states now? Explain.

So do you think we made the system better by taking a seat from Rhode Island and giving it to Virginia? Well, George Washington and Thomas Jefferson would say yes, and not just because they were Virginians.

Jefferson's Apportionment Method

When President Washington vetoed the bill approving Hamilton's method for the 1794 apportionment, Congress did not have enough votes to override it. So instead they passed a bill approving an apportionment method proposed by Thomas Jefferson.

Jefferson's method is a **divisor method**. To see what this means, consider again the calculations used to find standard quotas. For example, with the population figures shown in Table 11.1 and a total of 105 seats to be apportioned, the standard quota for Connecticut is

$$\frac{237{,}655}{3{,}893{,}874} \times 105 = 6.408,$$

and the standard quota for Delaware is

$$\frac{59{,}096}{3{,}893{,}874} \times 105 = 1.594.$$

Note that the only input numbers different in these two calculations are the populations of the states (the top numbers in the fractions on the left). In fact, the standard quota for each state could be calculated in the same way—by dividing the state's population by 3,893,874 (the total population of all the states) and then multiplying the result by 105 (the total number of seats). The only quantity that would vary from one calculation to the next is the population of the state itself. For this reason, standard quota calculations are often expressed in a slightly different, but mathematically equivalent, form. For example, Connecticut's standard quota calculation can be expressed as

$$\frac{237{,}655}{\frac{3{,}893{,}874}{105}} = 6.408,$$

and Delaware's as

$$\frac{59{,}096}{\frac{3{,}893{,}874}{105}} = 1.594.$$

Expressing these calculations in this way makes them a bit harder to follow. But this alternative form is useful in that it allows us to more easily identify the role of the divisor $\frac{3{,}893{,}874}{105}$ (which is 37,084.51) in each state's standard quota calculation. This value is called the **standard divisor**, and is completely determined by the total population of the system and the total number of seats to be apportioned, neither of which would change if the calculation were done for a different state. So, once the standard divisor for a system has been found, the standard quota for each state can be found by simply dividing the state's population by the standard divisor.

Question 11.10. In the apportionment from Question 11.3, use the standard divisor to find the standard quotas for all of the states.

The reason Jefferson's method is called a divisor method is because it works by modifying the standard divisor until a modified divisor results in modified quotas that give away exactly the correct number of seats when they are all rounded according to the same common convention. The specific details of Jefferson's method are as follows:

Jefferson Step 1: Find the standard quota for each state.

Jefferson Step 2: Round each standard quota down, and check to see if the sum of the rounded standard quotas equals the total number of seats to be apportioned. If so, the method is complete. Otherwise, continue with Step 3.

Jefferson Step 3: Choose a modified divisor that is different from the standard divisor, and use this modified divisor to calculate modified

quotas for each state (by dividing each state's population by the modified divisor).

Jefferson Step 4: Round each modified quota down, and check to see if the sum of the rounded modified quotas equals the total number of seats to be apportioned. If so, the method is complete. Otherwise, repeat Steps 3 and 4 (with a different modified divisor) until the method is complete.

Note that steps 3 and 4 in Jefferson's method include some inherent trial and error, a fact that has the potential to make Jefferson's method significantly more time-consuming than Hamilton's. Jefferson's method, however, does have one clear advantage—it rounds all of the quotas according to the same common convention. This is unlike Hamilton's method, in which a particular decimal part might in some instances be rounded up, but in other instances be rounded down. In fact, the consistency of the rounding convention employed by Jefferson's method is likely what caused Washington and Jefferson to believe it was fairer than Hamilton's.

Question 11.11.*

(a) Use Jefferson's method to apportion 105 seats to the fifteen states with population figures shown in Table 11.1. Write down the apportionment that results from each modified divisor you try, including those that fail to give away exactly 105 seats.

(b) In your apportionment from part (a), which states were treated best, and which were treated worst?

Despite the fact that Hamilton's method is easier to use than Jefferson's, it was Jefferson's method that was used to apportion the seats in the House in 1794 (although, as we noted earlier, the population figures used in 1794 were slightly different from those shown in Table 11.1). If you completed Question 11.11 correctly, the apportionment you found using Jefferson's method should differ from the apportionment you found using Hamilton's method (in Question 11.3), but only in the fact that Jefferson's method should have caused Rhode Island to lose a seat to Virginia. So we see that although it is possible for Hamilton's and Jefferson's methods to yield identical apportionments for the same system, they don't have to. And in fact, with the slightly different population figures that were actually used in 1794, Hamilton's method gave both Delaware and Rhode Island a second seat, and it was Delaware that lost its second seat to Virginia under Jefferson's method, leaving Rhode Island still with two.

What's really important to note about this is that in both the final apportionment from Question 11.11 and the actual apportionment from 1794, a small state lost a seat to a large state when compared to what the results would have been for the same system under Hamilton's method. This phenomenon actually occurs quite regularly under Jefferson's method, and the next question and subsequent discussion reveals why.

Question 11.12. Explain why in step 2 of Jefferson's method, too few seats will always be given away (except in the essentially impossible event that the original standard quotas are all exact whole numbers).

Thus, after step 2 of Jefferson's method, more seats will always need to be given away. As a result, in Jefferson's method, the standard divisor will always need to be modified lower, since lowering the divisor will increase the quotas (because we will be dividing by a smaller number). However, modifying a divisor lower will cause larger quotas to increase more quickly than smaller ones. For example, notice how much bigger the modified quotas for Rhode Island and Virginia are in Question 11.11 than their respective standard quotas. (Virginia's modified quota should have increased from its standard quota much more than Rhode Island's.)

So, in general, larger quotas have a greater chance than smaller quotas of increasing over the next whole number when the divisor is lowered. Thus, states with larger quotas (and correspondingly larger populations) have a better chance than states with smaller quotas of receiving additional seats under Jefferson's method. In fact, if one state's quota is significantly larger than those of other states (as Virginia's was in 1790), the larger quota could increase all the way over the next *two* whole numbers before all of the additional seats have been given away. The next question illustrates this phenomenon.

Question 11.13. The census of 1820 recorded populations of 1,368,775 for the state of New York and 8,969,878 for the entire U.S. Based upon the populations of New York and the other states recorded in this census, 213 seats in the House were to be apportioned in 1822.

(a) Using the 1820 census data, calculate the standard divisor and New York's standard quota.

(b) In the apportionment of 1822, Jefferson's method was used with a modified divisor of 39,900. Using this modified divisor, find New York's modified quota and the final number of seats the state was given. Do you think it was fair that New York received this number of seats? Why or why not?

The apportionment of 1822 revealed a serious flaw in Jefferson's method, as we saw in part (b) of Question 11.13. Unfortunately, though, Congress did nothing in response. Perhaps it was believed that the problem was a fluke, an anomaly that would not occur again or at least not occur often enough to be a real cause for concern. But then the same thing occurred in the very next apportionment, in 1832 (based on data from the census of 1830), when Jefferson's method gave New York 40 seats in the House even though its standard quota was only 38.59.

At this point, the problem had to be addressed. Daniel Webster, among many others, was outraged. Webster, an accomplished orator, in one of his

more famous speeches argued vehemently before Congress that apportioning 40 seats to New York was not only troublesome but unconstitutional.

Two alternative apportionment methods were subsequently presented to Congress. One was proposed by John Quincy Adams and is identical to Jefferson's method, except that in steps 2 and 4 of Adams' method, quotas are rounded up rather than down.

Question 11.14.*

(a) Use Adams' method to apportion 105 seats to the fifteen states with population figures shown in Table 11.1. Write down the apportionment that results from each modified divisor you try, including those that fail to give away exactly 105 seats.

(b) In your apportionment from part (a), which states were treated best, and which were treated worst?

Question 11.15. In step 3 of Adams' method, how will the standard divisor always need to be modified: higher or lower? Give a convincing argument to justify your answer.

Since Adams' method is just the mirror image of Jefferson's, it is obviously no better. Just as Jefferson's method works consistently and unfairly in favor of larger states, Adams' method works consistently and unfairly in favor of smaller states. So Adams' method was never actually used to apportion the seats in the U.S. House. But we will give Adams the benefit of the doubt and suppose that maybe he just carried to an extreme level our Founding Fathers' desire to provide protection for the smaller states in the Union.

The second apportionment method presented to Congress as an alternative to Jefferson's method was proposed by Daniel Webster himself. We'll investigate Webster's method in the next section, but before we do so, let's first formalize one of our observations about Jefferson's method.

As we noted, New York was given 40 seats in the 1832 apportionment, even though its standard quota was only 38.59. Using current terminology, we would say that this violates the **quota rule**. The quota rule states that, in an apportionment, each state should be given a number of seats equal to its original standard quota, rounded either up or down. An apportionment method for which the quota rule always holds is said to *satisfy quota*. Jefferson's method *violates quota* because it is possible for a state to receive more seats than its standard quota rounded up. (Adams' method also violates quota, as it is the mirror image of Jefferson's.) Moreover, it is actually very common for Jefferson's method to violate quota. As it turns out, if Jefferson's method had continued to be used, every apportionment of the House since 1852 would have violated quota.[3]

[3] After the first apportionment of the House in 1794, the seats were then supposed to be reapportioned every decade on years ending in 2, using the population figures recorded in the national census from two years prior.

However, 1832 was the end of Jefferson's method, which brings us back to Daniel Webster.

Webster's Apportionment Method

The apportionment method proposed by Daniel Webster in 1832 was, like Adams' method, a divisor method that differed from Jefferson's only in the rule used for rounding. In particular, Webster's method used conventional rounding (instead of always rounding down) in steps 2 and 4.

Question 11.16.*

(a) Use Webster's method to apportion 105 seats to the fifteen states with population figures shown in Table 11.1. Write down the apportionment that results from each modified divisor you try, including those that fail to give away exactly 105 seats.

(b) In your apportionment from part (a), which states were treated best, and which were treated worst?

Question 11.17. Clearly explain why in step 3 of Webster's method, the standard divisor could need to be modified either higher or lower.

Webster's method was used for the apportionment of 1842, and it is regarded by many modern-day experts as the best of all apportionment methods. One reason for this high regard is the fact that Webster's method is completely neutral in how it treats larger states in comparison to smaller states. In more precise terms, Webster's method slightly favors smaller states when conventional rounding of the standard quotas gives away too many seats, and larger states when conventional rounding of the standard quotas gives away too few seats. But since conventional rounding of standard quotas is equally likely to give away too many seats as too few, the method is ultimately neutral.

It has been shown that Webster's method (and, indeed, any divisor method) can violate quota, but examples of such violations tend to be so contrived that they would occur only very rarely in real-life situations. In fact, if Webster's method had been used consistently from the first apportionment of the House in 1794 to the most recent reapportionment in 2012, it would still have yet to violate quota even once.

Even so, the very possibility of violating quota left Congress leery of Webster's method, especially after the fiasco of 1832. So, in 1850, Congressman Samuel Vinton proposed what he believed to be a brand new apportionment method. As it turned out, it was identical to Hamilton's method. In Vinton's defense, nobody else remembered it either. So Congress called the new method Vinton's method and, since it can never violate quota, received it warmly. (To avoid confusion, we'll continue to call it Hamilton's method.)

Question 11.18. Clearly explain why Hamilton's method can never violate quota.

In 1852, Congress passed a law adopting Hamilton's method as the official method of apportionment for the seats in the House. However, since in practice Hamilton's and Webster's methods often yield identical apportionments, an unofficial compromise was also adopted that in 1852 and future years, Congress would have a total number of seats in the House for which Hamilton's and Webster's methods would yield identical apportionments. And since members of Congress do not typically like to legislate themselves out of business, we could better describe this compromise as saying that in 1852 and future years, Congress would *increase* the total number of seats in the House to a number for which Hamilton's and Webster's methods would yield identical apportionments.

This agreement lasted only until 1872, when, in direct violation of the Constitution (which specifies that some prescribed method must be used to apportion the seats in the House) and illegally (violating the 1852 law designating Hamilton's method as the official method), Congress passed an apportionment bill that wasn't based on any method at all. The apportionment gods got their revenge in the presidential election of 1876, when Rutherford B. Hayes defeated Samuel Tilden thanks to Electoral College numbers resulting from the unconstitutional apportionment of 1872. If either Hamilton's or Webster's method had been used to apportion the seats in the House in 1872, Tilden would have easily won the election. Undoubtedly with great humility, Congress went back to Hamilton's method in 1882, which is where this story takes yet another unexpected turn.

Three Apportionment Paradoxes

As part of the procedure for apportioning the seats in the House of Representatives in 1882, the U.S. Census Bureau supplied Congress with a table showing the apportionments under Hamilton's method for all sizes of the House between 275 and 350 seats. This table revealed something truly bizarre.

With a House size of 299 seats, Alabama's standard quota was 7.646, Illinois' was 18.640, and Texas's was 9.640. A ranking of the 38 states at the time, starting with the one whose standard quota had the largest decimal part and ending with the one whose standard quota had the smallest, placed Alabama 20th. In addition, with a House size of 299 seats, there were exactly 20 surplus seats to be given away in Hamilton's method.

With a House size of 300 seats, the standard quotas for all of the states were naturally a bit larger. Alabama's increased to 7.671, Illinois' to 18.702, and Texas's to 9.672. This placed Illinois 20th in the new ranking of the states, with Texas next in line. And with a House size of 300 seats, there were 21 surplus seats to be given away instead of just 20.

Question 11.19. Based on the figures in the previous two paragraphs, how many seats would Hamilton's method have given to Alabama, Illinois, and Texas if the total size of the House in 1882 had been 299 seats? How many seats would each state have received if the total size of the House had been 300 seats? Does anything about these two potential apportionments strike you as being unusual or unfair? Explain.

The remarkably unfair phenomenon you observed in Question 11.19 is actually a relatively common occurrence under Hamilton's method when the states involved have highly varied populations. When it was observed in 1882, it finally validated George Washington's veto of Hamilton's method from no less than 88 years prior.

This unfair phenomenon also provides us with another example of a paradox, but in a completely different setting from what we've seen before. Just as Condorcet's voting paradox from Chapter 3 seems to defy logic in the voting world, in the apportionment world it contradicts common sense that increasing the number of seats in the House from 299 to 300 would cause Alabama to *lose* a seat. This apportionment paradox (when increasing the number of seats in an apportioned system, in and of itself, causes a state to lose a seat) is commonly referred to as the **Alabama paradox**.

So how did Congress resolve the paradox in 1882? They opted to go with a House size of 325 seats, a value for which the paradox did not present itself. Then they crossed their fingers and hoped the paradox would never be heard from again. And then 1902 happened.

Question 11.20. As part of 1902 apportionment, the U.S. Census Bureau supplied Congress with a table showing the apportionments under Hamilton's method for all sizes of the House between 350 and 400 seats. When the number of seats in the House was 350-382, 386, 389, and 390, Maine would be given three seats, but for all other House sizes Maine would be given four seats. Also, when the number of seats in the House was 357, Colorado would be given two seats, but for all other House sizes Colorado would be given three seats. Based only on these figures for Maine and Colorado, would the Alabama paradox have occurred in 1902 for at least one House size between 350 and 400 seats (assuming Hamilton's method was used)? If so, for which House sizes and which states?

Naturally then, a bill was presented in Congress to apportion the seats in the House in 1902 using Hamilton's method with a House size of 357 seats. (We'll pause for a moment to let that sink in.) As you might suspect, the resulting debate got a little heated. In the end, the bill was defeated, and Hamilton's method was scrapped once and for all. The final 1902 apportionment was done using Webster's method with a House size of 386 seats.

The would-be occurrence of the Alabama paradox in 1902 was the final death blow for Hamilton's method, and it has never again been used to apportion the seats in the House. However, scholars have continued to study

Hamilton's method, and two additional paradoxes that can occur under the method have been discovered.

One, called the **population paradox**, was also discovered around the time of the 1902 apportionment, when it was observed that Hamilton's method would have caused Virginia to lose a seat to Maine, even though the population of Virginia had grown by a larger percentage over the previous decade than that of Maine.

Question 11.21.* According to the U.S. Census Bureau, the population of Nevada grew from 1,998,257 in 2000 to 2,700,551 in 2010, and the population of Illinois grew from 12,419,293 in 2000 to 12,830,632 in 2010. In the 2012 apportionment, which was based on the Census Bureau's population figures from 2010, Nevada gained a seat in the House while Illinois lost a seat. Based on these figures for Nevada and Illinois, can you conclude that the population paradox occurred in 2012? Why or why not?

Question 11.22. According to the U.S. Census Bureau, the population of Nevada grew from 1,201,598 in 1990 to 1,998,257 in 2000, and the population of Illinois grew from 11,435,813 in 1990 to 12,419,293 in 2000. In the 2002 apportionment, which was based on the Census Bureau's population figures from 2000, Nevada gained a seat in the House while Illinois lost a seat. Based on these figures for Nevada and Illinois, can you conclude that the population paradox occurred in 2002? Why or why not? And why is this question more interesting than Question 11.21?

Another paradox that can occur under Hamilton's method was discovered in 1907, when Oklahoma joined the Union as the 46th state. Because it was not yet time for the next reapportionment, Congress decided to simply increase the size of the House and give Oklahoma a number of seats proportional to its population. This decision resulted in 5 seats being added to the House, increasing its total size from 386 to 391. However, it was then noted that if Hamilton's method had been used to apportion 386 seats to the 45 states in 1902, and then again to apportion 391 seats to the 46 states in 1907, New York would have lost a seat to Maine! In other words, adding a new state and its fair share of seats would have caused changes (both positive and negative) in the number of seats given to other existing states. Incidents such as this are examples of what is now commonly called the **new-states paradox**.

The discovery of the population and new-states paradoxes supported the decision by Congress in 1902 to stop using Hamilton's method. And these paradoxes, as well as the Alabama paradox, will never bother Congress again, as it turns out they are impossible under divisor methods such as Jefferson's, Adams', and Webster's. (See Question 11.32.)

Hill's Apportionment Method

Although, at this point, our history lesson has only brought us as far as 1907, we're actually nearing the end of the apportionment story. We have

but a single new method left to discuss—the one currently used to apportion the seats in the U.S. House of Representatives. As we mentioned in the previous section, Webster's method was used for the 1902 apportionment. For the 1912 apportionment, Walter Willcox, the first American to investigate apportionment from a theoretical perspective, argued successfully to Congress that Webster's method should again be used. Around this same time, Joseph Hill, Chief Statistician of the U.S. Census Bureau, proposed a new method—with a strong endorsement from famed mathematician Edward Huntington.

Hill's method is yet another divisor method, and it is almost exactly identical to Webster's. The one lone difference between Webster's and Hill's methods is in the location of the cutoffs used for rounding quotas. Under Webster's method, a quota is rounded up if its decimal part is bigger than or equal to .50, and down otherwise. We can view this .50 cutoff as the decimal part of the average, or *arithmetic mean*, of the two whole numbers closest to the quota. For example, under Webster's method the quota 5.482 would be rounded down because its decimal part is smaller than the decimal part of 5.50, which is the arithmetic mean of 5 and 6.

Hill's method, instead of using an arithmetic mean to determine the cutoff for rounding a quota up or down, uses the *geometric mean* of the two whole numbers closest to the quota. The geometric mean of any two whole numbers x and y is simply $\sqrt{x \cdot y}$. So, for Hill's method, a quota is rounded up if its decimal part is bigger than or equal to the decimal part of the geometric mean of the two closest whole numbers, and down otherwise. For example, for the quota 5.482, the geometric mean of the two closest whole numbers is $\sqrt{5 \cdot 6} = 5.477$. So, because the decimal part of 5.482 is bigger than the decimal part of 5.477, under Hill's method the quota 5.482 would be rounded up to 6. But (and notably!), since $\sqrt{15 \cdot 16} = 15.492$, under Hill's method the quota 15.482 would be rounded down to 15.

Question 11.23.*

(a) Use Hill's method to apportion 105 seats to the fifteen states with population figures shown in Table 11.1. Write down the apportionment that results from each modified divisor you try, including those that fail to give away exactly 105 seats.

(b) In your apportionment from part (a), which states were treated best, and which were treated worst?

(c) How does the apportionment you found using Hill's method in part (a) compare to those you found for this same system in previous questions using Hamilton's, Jefferson's, Adams', and Webster's methods?

Question 11.24. Is Hill's method biased in favor of larger states, smaller states, or neither? (Hint: See the last two sentences before Question 11.23.)

Both Webster's and Hill's methods were considered for the 1922 apportionment, and the two methods produced significantly different outcomes. By this time the number of seats in the House had been fixed by law, and so as a result of the discrepancy between Webster's and Hill's methods, no apportionment bill was passed. Consequently, the 1912 seat totals were held over without any reapportionment whatsoever. (This was, of course, another direct violation of the Constitution.)

In preparation for the 1932 apportionment, a committee of members from the National Academy of Sciences was formed to study Webster's and Hill's methods. The committee endorsed Hill's method, a powerful victory in the contest between the two. But then, in a remarkable twist of fate, Webster's and Hill's methods produced identical apportionments using the 1930 census data. So proponents of either method could claim that theirs was the one used in 1932.

For the 1942 apportionment, Webster's and Hill's methods came very close to again producing identical apportionments. The only difference between the two was that Hill's method gave a single extra seat to Arkansas at Michigan's expense. At the time, Michigan tended to elect Republican legislators, while Arkansas tended to elect Democrats. The vote on the resulting apportionment bill split exactly along party lines, with the Democrats supporting Hill's method and the Republicans supporting Webster's. Because the Democrats had the majority, it was Hill's method that passed through Congress. President Franklin D. Roosevelt, also a Democrat, signed the method into "permanent" law, and it has been used for every reapportionment of the House since.

Another Impossibility Theorem

Both Webster's and Hill's methods are held in high regard by scholars who study apportionment. In fact, most modern-day experts are proponents of one of the two methods, and with good reason—both are relatively neutral in how they balance power between small and large states, and both are incapable of producing the three apportionment paradoxes we saw earlier.

With that said, both Webster's and Hill's methods can violate quota, which raises the question: Wouldn't it be nice if we could find an apportionment method that was incapable of producing paradoxes *and* would never violate quota? Well, in the 1970s, mathematicians Michel Balinski and Peyton Young set out to find such a system. The result of their search might have surprised us way back in Chapter 1, but not any longer.

Balinski and Young's Theorem. It is impossible for an apportionment method to always satisfy quota and be incapable of producing paradoxes.

The basic idea behind the proof of Balinski and Young's Theorem is actually quite simple. They first showed that the only apportionment methods that avoid the population paradox are divisor methods. But it was already

known before Balinski and Young that every divisor method is capable of violating quota. So in order to avoid the population paradox, you need a divisor method. And as soon as you have a divisor method, you run the risk of violating quota. Balinski and Young's Theorem follows easily from these two facts.

Question 11.25. In light of Balinski and Young's Theorem, which do you think is more important for an apportionment method: that it never violate quota or that it avoid the population paradox? (Remember—you can't have both!) Give a convincing argument to justify your answer.

Like Arrow's Theorem did in the world of voting, Balinski and Young's Theorem takes us full circle in the world of proportional representation; it shows us that, as was the case with voting, proportional representation is incapable of being free from controversy.

Even so, some apportionment methods are clearly better than others. For instance, Hamilton's method is certainly the easiest to use, and that is why, despite its drawbacks, it is still the method of choice in a number of countries around the world. On the surface, Webster's method seems to be the fairest from a mathematical perspective. But Hill's was the method endorsed by the National Academy of Sciences after it was carefully compared to Webster's in the late 1920s. It should also be noted that when Balinski and Young first presented a proof of their famed impossibility theorem in 1980, they went on to argue convincingly why they believed Webster's method was the best.

So, as with voting, we may never arrive at a definitive answer to the apportionment problem. Nevertheless, what we've learned in this chapter at least gives us the tools we need to analyze various apportionment methods and approach proportional representation from a reasoned and logical perspective.

Concluding Remarks

In this chapter, we've looked at five different methods for apportioning seats in a legislative body based on the populations of the states or districts to be represented. In each case, for the sake of space, the only example we considered was the initial apportionment of the seats in the U.S. House of Representatives. The apportionment methods we investigated can be applied in a great variety of situations, depending on whether we are interested in apportioning a fixed number of legislative seats to counties in a state, or board of supervisors seats to districts in a city, or even pieces of candy to children in a family. The methods we investigated work exactly the same way in any situation in which the objects to be distributed cannot be divided into smaller pieces.

If you wish to study apportionment in more detail, you may find yourself considering examples in other books or online. Be advised that, in other

sources, the five apportionment methods we discussed may be referred to by different names. For instance, Hamilton's method is sometimes called the method of largest remainders, Jefferson's the method of greatest divisors or d'Hondt's method (as it is known in Europe), Adams' the method of smallest divisors, Webster's the method of major fractions or the Webster-Willcox method, and Hill's the method of equal proportions or the Hill-Huntington method.

Questions for Further Study

Question 11.26. When the first national U.S. census was conducted in 1790, Maine was still considered part of Massachusetts. If Maine had been considered a separate state at the time, then the 105 seats in the initial 1794 apportionment of the House would have been distributed among sixteen states instead of fifteen. Assuming the population of Maine in 1790 was 96,643, use the population figures in Table 11.1 to recalculate the 1794 apportionment, viewing Maine as a separate state. Use at least two of the apportionment methods we looked at in this chapter, and write a summary comparing the resulting apportionments both to each other and to the apportionments you calculated in this chapter with only fifteen states. (Note: When you calculate a new apportionment, don't forget to adjust the population of Massachusetts by subtracting Maine's from it.)

Question 11.27. Remember the Marquis de Condorcet? Well, as it turns out, he proposed an apportionment method too. His method was a divisor method, and his convention for rounding was to round a number up if its decimal part was bigger than or equal to .40, and down otherwise. Was Condorcet's method biased in favor of larger states, smaller states, or neither, or is it impossible to determine this from the given information? Explain.

Question 11.28. Research Lowndes' apportionment method, and write a complete summary of your findings. Include in your summary a description of how the method works, at least two small numerical demonstrations of what "relative fractional parts" means, who first proposed the method and when this occurred, how the method compares to the other apportionment methods we looked at in this chapter, and—if you can determine so—whether the method can violate quota or produce any of the three apportionment paradoxes.

Question 11.29. Investigate Dean's apportionment method (also known as the method of harmonic means), and write a complete summary of your findings. Include in your summary a description of how the method works, at least two small numerical demonstrations of what the *harmonic mean* of two numbers is, who first proposed the method and when this occurred, how the method compares to the other apportionment methods we looked

at in this chapter, and—if you can determine so—whether the method can violate quota or produce any of the three apportionment paradoxes.

Question 11.30. Find out which countries around the world use Hamilton's method to allocate seats in their governmental legislative bodies. Then investigate which countries use Jefferson's method.

Question 11.31. Which of the apportionment methods from this chapter (including the ones from Questions 11.27–11.29, if you did those questions) do you think is best, and why? Give a convincing argument to justify your answer.

Question 11.32. Why can divisor methods not produce any of the three apportionment paradoxes we looked at in this chapter? Either form your own explanation, or do some research and write a complete summary of your findings.

Question 11.33. Write a short biography of Walter Willcox, including his most important contributions both inside and outside of apportionment.

Question 11.34. Write a short biography of Edward Huntington, including his most important contributions both inside and outside of apportionment.

Question 11.35. Write short biographies of Michel Balinski and Peyton Young, including their most important contributions both inside and outside of apportionment.

Question 11.36. Find a copy of Article 1, Section 2 of the U.S. Constitution, and write a summary of what it states regarding apportionment. Then critique this section of the Constitution by identifying any deficiencies present in its description of how the seats in the U.S. House of Representatives should be apportioned to the states.

Question 11.37. Find a copy of George Washington's veto message when he vetoed Hamilton's apportionment method, and write a summary of what he stated. Then critique Washington's veto message. Do you think Washington's veto was personally motivated, or do you think he really saw some deficiencies in Hamilton's method?

Question 11.38. Find a copy of Daniel Webster's speech to Congress in which he argued that apportioning 40 seats to New York in 1832 was unconstitutional. Write a summary and critique of Webster's speech, evaluating his argument in light of what you learned in this chapter. Do you think Webster presented his case in the best possible way, or could his argument have been stronger?

Question 11.39. Find out exactly how the seats in the U.S. House of Representatives were apportioned in 1872, and write a complete summary of your findings.

Question 11.40. Investigate the results of the apportionments of the seats in the U.S. House of Representatives in 2002 and 2012, and write a summary comparing the two. Which states gained seats and which lost seats between 2002 and 2012? Do you think your state was treated fairly in the 2012 apportionment? Based on its population, is your state currently underrepresented, overrepresented, or perfectly represented in the House?

Question 11.41. In 1991, a lawsuit, *Montana* v. *United States Department of Commerce*, was filed in U.S. Federal District Court. Find out the details of this lawsuit and the subsequent Supreme Court ruling. Write a summary of your findings, including the outcomes of both the original lawsuit and the Supreme Court ruling, and your own personal feelings about what the outcomes should have been.

Question 11.42. Find an article in a popular media source that expresses a positive view of the method currently used to apportion the seats in the U.S. House of Representatives. Write a summary and critique of the article based on what you learned in this chapter.

Question 11.43. Find an article in a popular media source that expresses a negative view or questions the constitutionality of the method currently used to apportion the seats in the U.S. House of Representatives. Write a summary and critique of the article based on what you learned in this chapter.

Question 11.44. If Hamilton's method had been used for the 2012 apportionment of the House, would this have made a difference in the outcome of the 2016 U.S. presidential election? Completely explain your answer.

Question 11.45. If Jefferson's method had been used for the 2012 apportionment of the House, would this have made a difference in the outcome of the 2016 U.S. presidential election? Completely explain your answer.

Question 11.46. If Hamilton's method had been used for the 1992 apportionment of the House, would this have made a difference in the outcome of the 2000 U.S. presidential election? Completely explain your answer.

Question 11.47. If Jefferson's method had been used for the 1992 apportionment of the House, would this have made a difference in the outcome of the 2000 U.S. presidential election? Completely explain your answer.

Question 11.48. Create a spreadsheet in Microsoft Excel or similar software to implement one or more of the apportionment methods we looked at in this chapter. (The only special Excel functions you should need are `ROUNDDOWN` for Hamilton's and Jefferson's methods, `ROUNDUP` for Adams', `ROUND` for Webster's, and `SQRT` for Hill's.) Then use your spreadsheet to apportion the U.S. House for some year in which seats were apportioned.

Question 11.49. In 1991, a lawsuit, *Commonwealth of Massachusetts* v. *Mosbacher*, was filed in Massachusetts District Court. Find out the details

of this lawsuit, and write a summary of your findings. Include in your summary the outcome of the lawsuit and your own personal feelings about what the outcome should have been.

Question 11.50. Find a copy of the article "Outcomes of Presidential Elections and the House Size," by Michael G. Neubauer and Joel Zeitlin, in the journal *PS: Political Science & Politics*. Write a summary of what the article states regarding the relationship between the apportionment of the U.S. House of Representatives and U.S. presidential elections.

Answers to Starred Questions

11.3. There should be a total of 9 surplus seats. In the correct apportionment, Connecticut should be given 6 seats, Delaware 1 seat, Maryland 9 seats, and Virginia 20 seats.

11.4. See the first paragraph after Warmup 11.1 for these percentages.

11.6. Delaware has one representative and a population of 59,096. Thus each of Delaware's representatives represents on average $\frac{59{,}096}{1} = 59{,}096$ residents. The calculations for the other states can be done similarly (and are perhaps more interesting).

11.9. With the revised numbers, Virginia is the fourth-best treated state, and Rhode Island is the worst (even worse than Delaware!).

11.11. A modified divisor of 35,000 produces the correct apportionment. (There are other modified divisors that produce the correct apportionment as well. Typically with divisor methods, there are multiple divisors that all give away exactly the correct numbers of seats.) The correct apportionment here should be identical to the one from Question 11.3, except that Rhode Island should have lost a seat to Virginia.

11.14. A modified divisor of 39,600 produces the correct apportionment. In comparing the correct apportionment here and the one from Question 11.3, four states should have gained a seat and four should have lost a seat.

11.16. A modified divisor of 37,616 produces the correct apportionment. The correct apportionment here should be identical to the one from Question 11.3, except that Maryland should have lost a seat to Delaware.

11.21. The population of Nevada grew by $\frac{2{,}700{,}551-1{,}998{,}257}{1{,}998{,}257} = 35.1\%$ between 2000 and 2010, but during this same time period the population of Illinois grew by only $\frac{12{,}830{,}632-12{,}419{,}293}{12{,}419{,}293} = 3.3\%$. So we cannot conclude that the population paradox occurred.

11.23. A modified divisor of 37,670 produces the correct apportionment. The correct apportionment here should be identical to the one from Question 11.3, except that North Carolina should have lost a seat to Delaware.

Chapter 12

Choosing Your Voters

Focus Questions

In this chapter, we'll explore the following questions:

- What is gerrymandering, and what are some recent examples of gerrymandering in the United States?
- What are some of the laws and regulations that govern redistricting?
- What is compactness, and how is it related to gerrymandering? What are some different ways of measuring the compactness of a congressional district?
- What are some ways to measure the partisan fairness of a districting plan?
- What is the efficiency gap, and what kind of information does it provide about partisan fairness?
- What are some possible solutions to the gerrymandering problem?

Warmup 12.1. In Figure 12.1, suppose that each circle represents a voter, with the filled circles representing Republicans and the unfilled circles representing Democrats.

(a) If you had to divide the population into five congressional districts, each having an equal number of voters, how would you do it? Where would you draw the district boundaries?

(b) In the plan from your answer to part (a), how many districts would have a majority of Republican voters, and how many would have a majority of Democrats?

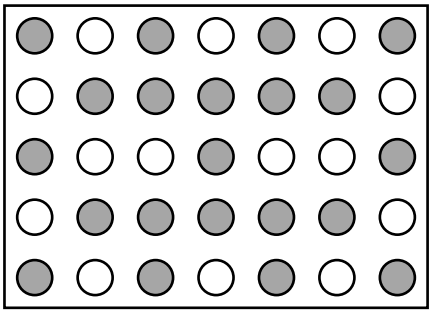


FIGURE 12.1. A redistricting example

(c) Suppose you are a Republican who wants to make sure that your party has a majority in as many districts as possible. Where would you draw the district boundaries, and how many districts would your party control?

(d) Repeat part (c), but this time assume that you want to maximize the number of districts in which Democrats have a majority.

As we learned in Chapter 11, changes in the U.S. population, as measured by each decennial census, may lead to states gaining or losing seats in the House of Representatives. In addition, shifts in population may cause congressional districts to have unequal numbers of residents. Therefore, each census brings an opportunity (and in most cases a legal obligation) to redraw congressional district boundaries to create an appropriate number of equal-population districts. This redistricting process seems simple enough on the surface. However, as we saw in Warmup 12.1, there are often many ways to draw the congressional districts in a state, and some districting plans may seem fairer than others.

In the example from Warmup 12.1, 60% of the voters were Republicans, and 40% were Democrats. Since there are five districts in total, you might expect that any reasonable division of the state into districts would result in three districts where Republicans hold a majority and two where Democrats are in control—and, of course, there is a way to draw the district boundaries so that this happens. But even some of the simplest solutions do not result in this outcome. For example, using four horizontal lines to divide the state into five rectangular districts results in a 4-1 split between Republican and Democratic districts. It's also possible to draw the boundaries so that Republicans control all five districts. There is an argument to be made for each of these outcomes; in fact, since Republicans do have a fairly substantial majority overall in the state, it makes sense for the state's congressional delegation to be mostly—or even entirely—Republican. But what if that didn't happen? What if, in spite of support from only 40% of the voters, Democrats won three of the five congressional districts? In fact, this is possible with a clever drawing of the district boundaries. Perhaps you even discovered a way to do it in part (d) of Warmup 12.1. If not, look back at

Figure 12.1 and try again. Here's a hint: For Democrats to win more seats than they deserve, they'll need to make the most of their votes, winning as many seats as possible by razor-thin margins. Republicans are going to win at least two districts regardless of how the boundaries are drawn (do you see why?), so Democrats will be best served if the Republicans win those two districts by unnecessarily wide margins, with the remaining Republican votes spread out among the other three districts.

Gerrymandering

As we've seen, the outcome of congressional redistricting depends not only on the distribution of the voters, but also on where the district boundaries are drawn. The potential to manipulate the system in favor of one party or another is significant. In fact, President Barack Obama put it this way in his book, *The Audacity of Hope* ([**38**], p. 103):

> *These days, almost every congressional district is drawn by the ruling party with computer-driven precision to ensure that a clear majority of Democrats or Republicans reside within its borders. Indeed, it's not a stretch to say that most voters no longer choose their representatives; instead, representatives choose their voters.*

In light of this observation, it shouldn't surprise us too much to see congressional districts like the ones shown in Figures 12.2–12.4.

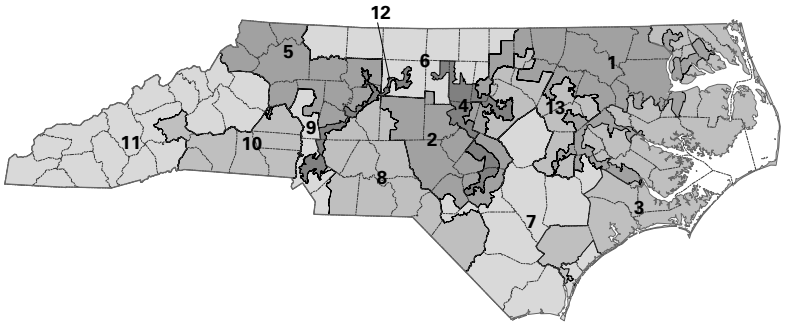


FIGURE 12.2. North Carolina congressional districts, 2013–2016

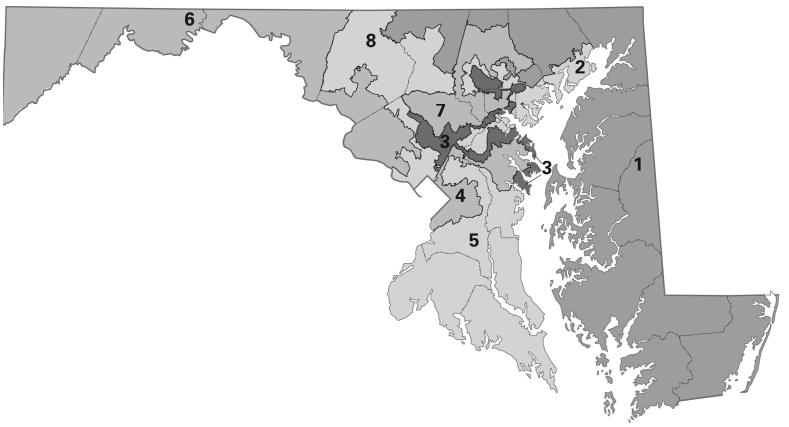


FIGURE 12.3. Maryland congressional districts, as of 2013

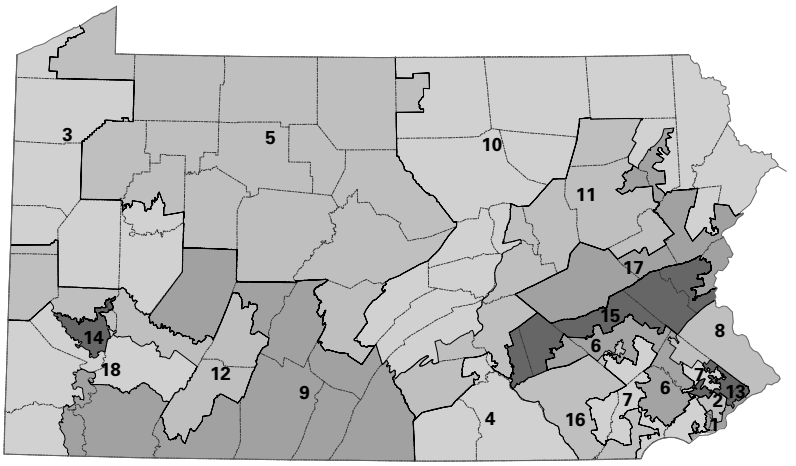
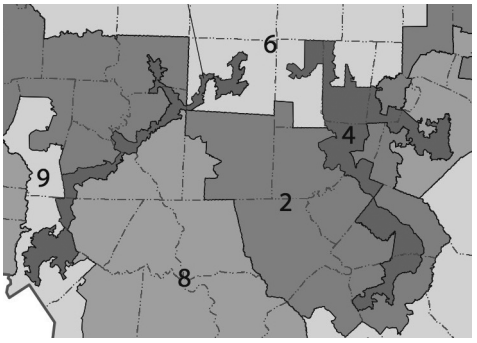


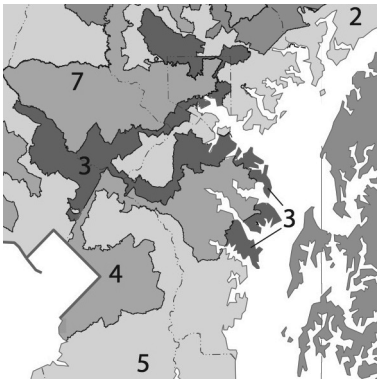
FIGURE 12.4. Pennsylvania congressional districts, 2013–2018

Notice that in each of these examples, several congressional districts have been drawn with strange, convoluted shapes. If we zoom in on a few areas, as shown in Figure 12.5, we can get an even better look. In each of these cases, districts have been drawn in a way that gives an advantage to one party or the other by concentrating like-minded voters into non-competitive districts and/or dividing like-minded voters among multiple districts, thereby diluting their power. These techniques are called *packing* and *cracking*, respectively.

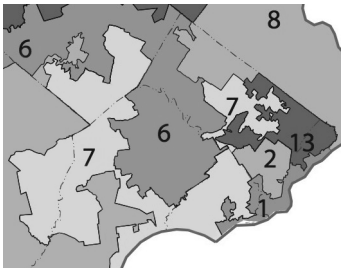
Question 12.2.* In the districting plan shown in Figure 12.6, Republicans earn only 12 of the 25 votes, but win 4 of the 5 seats. Explain how this plan uses packing and cracking to give Republicans an advantage.



Winston-Salem, Charlotte, Greensboro, Raleigh, Durham



Baltimore/Annapolis region



Philadelphia region

FIGURE 12.5. Redistricting closeups

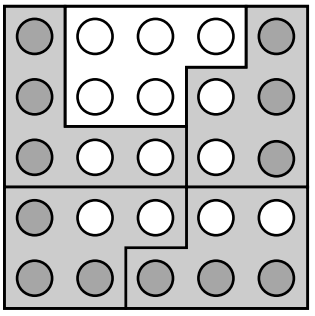


FIGURE 12.6. An example of packing and cracking

Oddly shaped congressional districts, like those in Figures 12.2–12.6, are often viewed as evidence of *gerrymandering*, which is the practice of drawing congressional districts to give an advantage to a political party or class of voters. The word *gerrymandering* was coined in 1812 when Massachusetts Governor Elbridge Gerry passed a bill to redraw the State Senate election districts in order to favor his party. The resulting map included a district that looked to some like a salamader. The *Boston Weekly Messenger* subsequently printed an editorial cartoon, shown here in Figure 12.7, that stylized the district in question as a mythical dragon, dubbed the "Gerry-Mander."

FIGURE 12.7. The "Gerry-Mander" from the *Boston Weekly Messenger*, 3/26/1812

While Elbridge Gerry may have been the first to engage in gerrymandering, he certainly was not the last. In fact, there have been numerous recent court cases alleging unconstitutional violations of voter rights as a result of gerrymandering. While a detailed discussion of the legal issues surrounding gerrymandering is beyond the scope of our investigations here, it is important to note the two main types of claims that are typically brought in redistricting cases.

First, courts have regularly ruled redistricting plans to be unconstitutional when district boundaries are drawn to disadvantage or marginalize a particular racial or ethnic group. The 2013–2016 redistricting of North Carolina is one of the more recent examples of an unconstitutional racial gerrymander. In *Cooper* v. *Harris* (2017), the U.S. Supreme Court ruled that the plan violated the Equal Protection Clause of the 14th Amendment by packing African American voters into the 1st and 12th districts. (See Figure 12.8.)

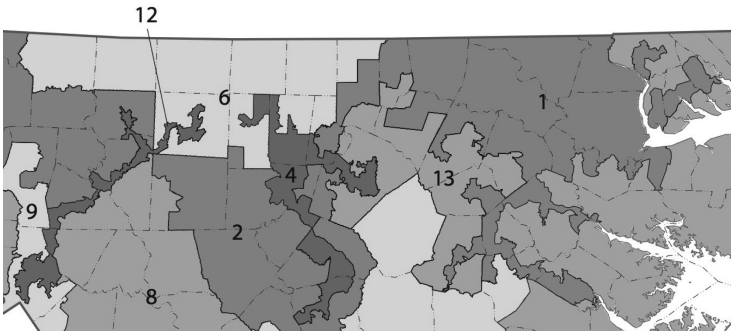


FIGURE 12.8. North Carolina's 1st and 12th districts, 2013–2016

But what about when districts are gerrymandered with partisan, but not racial, motivations? On this issue, the law remains unclear—or at least that was the case when this book was written. By the time you read it, the situation may have changed due to several recent and pending lawsuits. For example:

- In January 2018, the Pennsylvania Supreme Court ruled that the state's congressional districts violated the State Constitution by unfairly favoring Republicans. The U.S. Supreme Court declined to block the Pennsylvania court's order to redraw the districts, but this decision did not establish legal precedent for other cases since the original decision was based on Pennsylvania's *State* Constitution, rather than federal law.
- The Maryland map has been the subject of a protracted legal battle since 2013. In June 2018, the U.S. Supreme Court declined to issue an injunction that would have blocked the map from being used in the November 2018 congressional elections. Instead, the case was returned to the lower court for a trial.
- In 2015, a lawsuit was brought against the State of Wisconsin after the Republican-controlled legislature drew districts that led to Republicans winning 60 out of the 99 seats in the 2012 State Assembly election—in spite of receiving less than half of the vote statewide. In 2016, a three-judge panel ruled 2-1 that the plan was unconstitutional, which prompted an appeal by the State of Wisconsin to the U.S. Supreme Court. The Court ruled on the case, known as *Gill* v. *Whitford*, in June 2018. In their ruling, they "[expressed] no view on the merits of the plaintiffs' case" but instead argued that the plaintiffs had not established standing to sue because they had not yet proved "concrete and particularized injuries." In essence, the court held that the harm of vote dilution due to gerrymandering is "individual and personal in nature," and that to prevail, the plaintiffs would need to prove not only that Democrats in general had been harmed by the

districting plan, but that individual voters had been harmed specifically. The case was remanded to the lower court, giving the plaintiffs another opportunity to argue this more specific claim.

- After the previous redistricting of North Carolina was struck down due to racial gerrymandering, a new map was drawn—and contested on the basis of unconstitutional *partisan* gerrymandering. In January 2018, a three-judge panel ruled that the new map was unconstitutional and ordered that the districts be redrawn. In June 2018, one week after the *Gill* v. *Whitford* ruling, the U.S. Supreme Court threw out the lower court's decision and remanded the case for further consideration, specifically with regard to whether the plaintiffs had suffered the sort of direct and specific injury required by the *Gill* ruling. In August 2018, a three-judge panel ruled *again* that the North Carolina map was unconstitutional and that the plaintiffs—voting rights groups Common Cause and the League of Women Voters—did have standing to sue due to the fact that they had members in each of the contested districts. As of the writing of this book, it remains to be seen whether this most recent ruling will be appealed to the Supreme Court and, if so, whether the Court will accept the lower court's determination of standing and rule on the merits of the case.

What distinguishes these cases from previous ones is the fact that they are based entirely on allegations of partisan—not racial—gerrymandering. So that will be our primary focus in this chapter. And, of course, since this is a book on the *mathematics* of voting and elections, we'll look at several different ways in which mathematics can be used to identify gerrymandered districts.

Rules for Redistricting

Courts have held that the 14th Amendment to the U.S. Constitution requires congressional districts to have equal populations as much as is practical. Apart from that requirement and the requirements of the Voting Rights Act—which we'll discuss later—other rules for redistricting are determined on a state-by-state basis, normally by the state legislatures that are in charge of drawing the districts. A few common requirements include:

- **Contiguity**, meaning that all the parts of a congressional district must be physically connected in some way. This requirement can lead to some creative uses of unpopulated areas such as roads, bridges, and even exit ramps to connect parts of a district that would otherwise be geographically separated from each other.
- **Compactness**, meaning that districts should be. . . compact. If you think that's not a very good definition, you're right. Many states don't specifically define what compactness means, but the general

sense is that the residents in a district should live relatively close to one another, and that congressional districts should have relatively normal-looking shapes.

- **Preserving political boundaries**, meaning that, whenever possible, district boundaries should not cut through the middle of counties, cities, or other municipal entities.
- **Preserving communities of interest**, meaning that, whenever possible, district lines should not separate groups of voters who live near one another and have common political, social, or economic interests.

Geometry and Compactness

Although most state laws governing redistricting allow a lot of room for interpretation, the requirement of compactness is one that is particularly nebulous.

Question 12.3. The laws in the state of Idaho specify that "to the maximum extent possible, [redistricting plans] should avoid drawing districts that are oddly shaped." Using this incredibly precise definition of compactness, put the districts in Figure 12.9 in order from "least oddly shaped" to "most oddly shaped."

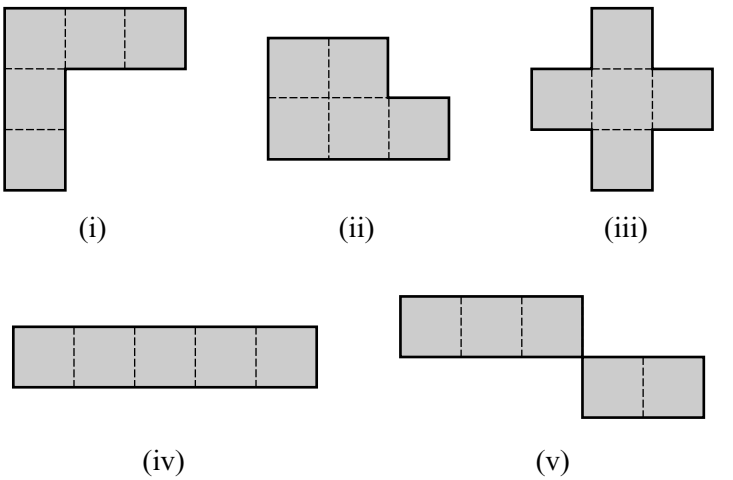


FIGURE 12.9. Some hypothetical district shapes

Question 12.4.* The state of Iowa considers compact districts to be those which are "square, rectangular, or hexagonal in shape, and not irregularly shaped, to the extent permitted by natural or political boundaries." When comparing two redistricting plans, Iowa state law prescribes two specific measures of compactness: length-width compactness and perimeter compactness.

(a) Length-width compactness assumes that a district is most compact when its length and width are equal. A district's compactness under this measure is the absolute value of the difference between its length and its width. Using this definition, order the districts in Figure 12.9 from most compact to least compact.

(b) Perimeter compactness assumes that the most compact districts are those with the shortest perimeter. (Recall that the perimeter of a shape is the length of its boundary.) Using this definition, order the districts in Figure 12.9 from most compact to least compact.

Question 12.5.* In the state of Michigan, districts are required to be "as compact as possible, measured by drawing a circle around the district, and assessing the area within the circle (and within the landmass of the state) but outside the district lines." In other words, the tighter a circle can be made to fit around the district, and the more the district fills up this circle, the more compact the district is. Using this definition of compactness, order the districts in Figure 12.9 from most compact to least compact. (Hint: All of the districts in Figure 12.9 have the same area, so Michigan's compactness measure will be completely determined by the size of the smallest circle that encloses each region. The diameter of this smallest circle will be the distance between the two points in the district that are the farthest away from one another. Or, if all else fails, just draw some circles and eyeball it.)

As you may have seen in Questions 12.3–12.5, different measures of compactness can tell different stories about which districts are more or less compact. There are a lot of other measures of compactness that we haven't looked at, but most of them measure characteristics such as indentation or convexity, jaggedness, and elongation or dispersal (how far the area of the district is dispersed from its center). In past rulings, courts have viewed odd shapes as "persuasive circumstantial evidence" of racial gerrymandering and have ruled maps unconstitutional on the basis of shape. However, there is no universally accepted standard for compactness—and at times no standard at all other than the "eyeball test" suggested by laws like those in Idaho. In addition, it's important to note that not every district that fails to meet compactness standards is the result of an unconstitutional gerrymander. This is where the law gets even murkier, with competing and sometimes even contradictory requirements.

As an example, consider Illinois' 4th congressional district, which consists of two separate neighborhoods in the Chicago area connected by a thin strip of Interstate 294, as shown in Figure 12.10.

Question 12.6. How does Illinois' 4th congressional district measure up using the definitions of compactness from Idaho, Iowa, and Michigan? Explain your answer.

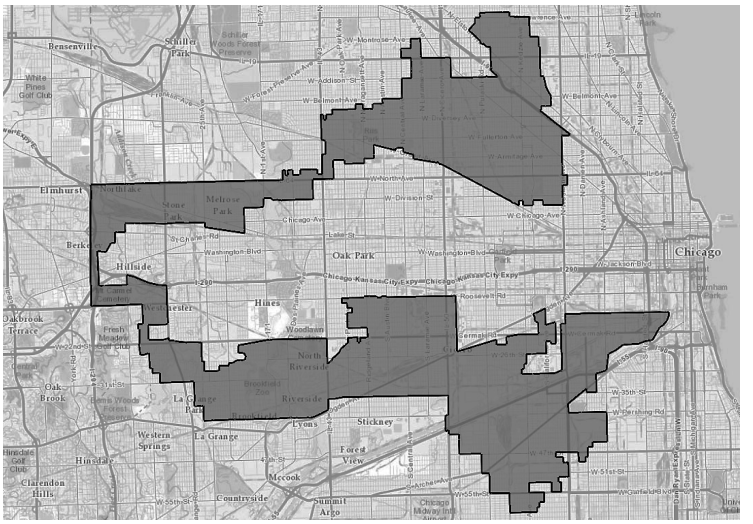


FIGURE 12.10. Illinois' 4th congressional district

At first glance, you may think that the strange shape of Illinois' 4th congressional district is clear evidence of gerrymandering, and in one sense you'd be right. This district *is* in fact gerrymandered, but not for nefarious purposes. In fact, it was created as a result of a lawsuit to join two Hispanic communities for the purpose of *strengthening* their voting power. In essence, Hispanic voters were *packed* into one district in order to prevent them from being *cracked* into two separate districts—an action that would have diluted their power, in violation of the 1965 Voting Rights Act.[1] So while courts have ruled that district lines cannot be drawn on the basis of race, the Voting Rights Act sometimes requires race to be a factor in order to create majority-minority districts that allow racial and ethnic minorities an opportunity to elect a candidate of their choosing. This tension is what some have referred to as the *Goldilocks rule*: Those charged with drawing district boundries must think about race and ethnicity—but not too much.

In the same way that strange, non-compact shapes should not be taken as conclusive evidence of unconstitutional gerrymandering, we must also be careful not to assume that a districting plan is fair simply because its districts are compact. The next question gives an example to illustrate this point.

Question 12.7. Consider a square state in which the voters are distributed as shown in Figure 12.11. Note that there are 72 Republican voters (represented by filled circles) and 90 Democrats (represented by unfilled circles).

[11] Interestingly, one of the attorneys originally involved in the creation of the district is reported to have said that they may have gone "a little too far," since the same representative, Congressman Luis Gutiérrez, has won the district in every election since 1992, capturing between 77 and 100 percent of the vote.

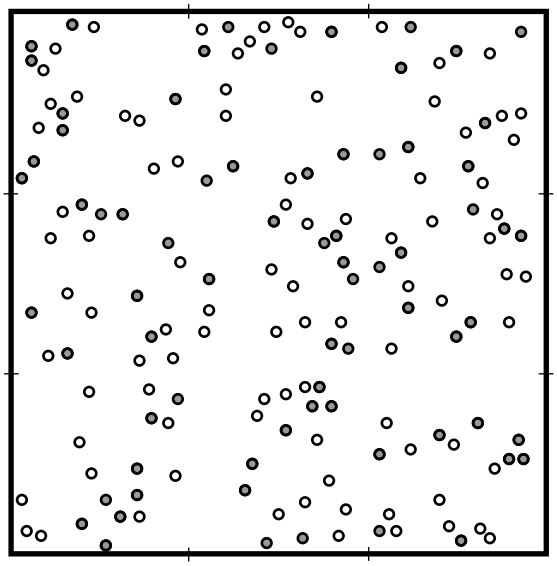


FIGURE 12.11. A square state

(a) If this state were to be divided into nine districts, and the number of seats won by each party was perfectly proportional to the number of votes they received, how many seats would each party win?

(b) Use the tick marks on the boundary of Figure 12.11 to divide the state into nine equal-sized, square-shaped districts. Using these districts, how many seats will each party win?

(c) Now try to draw nine equal-population districts that would result in each party winning the number of seats you specified in part (a).

(d) In which of your plans—part (b) or part (c)—were the districts more compact? Explain your answer.

Partisan Symmetry

Since compactness can't tell us the whole story—particularly when it comes to identifying partisan gerrymandering—we need to consider other ways of identifying when a districting plan gives an unfair advantage to one party or the other. One solution, which seems simple on the surface, would be to use proportional representation as an ideal standard. For example, we could say that if Republicans win 57% of the vote, then they should win 57% of the seats.

Question 12.8. Suppose that eight seats are up for grabs and Republicans win 57% of the vote. How many seats should they win?

Apart from the fact that achieving exact proportional representation may be numerically impossible—at least not without fractional seats—the U.S. Supreme Court has held that "the mere lack of proportional representation will not be sufficient to prove unconstitutional discrimination" (*Davis* v. *Bandemer*, 1986). The idea of a *winner's bonus*—that is, the winning party receiving more seats than they would be entitled to under a purely proportional system—is fairly widely accepted. The bigger question, at least in regard to fairness, concerns the notion of *symmetry*. To illustrate, suppose Republicans won 57% of the vote but won 75% of the seats. They earned a pretty hefty winner's bonus, but was the districting plan unfairly biased against Democrats? To answer this question, we could consider what would happen if the tables were turned and Democrats won 57% of the vote. Would they now win 75% of the seats? If so, the districting plan shows evidence of symmetry: Each party has a chance to receive a winner's bonus, but they do have to be a winner in order to get it. If, on the other hand, Republicans won only 43% of the vote but retained their winner's bonus—still winning more than half of the seats—then we would suspect that something was amiss.

In order to test a plan's symmetry, we need to consider hypothetical questions about what might happen if voters switched their votes from one party to the other. To answer these kinds of questions, we'll need to make assumptions about how the votes in each district change when the statewide vote totals change. The simplest and most common assumption is one called *uniform partisan swing*—meaning that in each district, the percentage of voters who change their votes from one party to the other is the same as the percentage of voters who switch statewide. So, for example, if 10% of Democratic voters *statewide* change their votes to the Republican candidate, then we'll assume that 10% of the Democrats *in each district* change their votes. In other words, we'll assume that the voters who switch from one party to the other are distributed proportionally among the various districts.

Question 12.9.* In the districting plan shown in Figure 12.12, assume that each circle represents 100 voters, with the filled circles representing Republican voters and the unfilled circles representing Democrats. Notice that Democrats earn 1300 of the 2500 votes statewide (52%) but win 4 out of the 5 districts (80%—a significant winner's bonus!).

(a) How many Democrats would need to switch their votes in order for Republicans to win 52% of the vote statewide? What percentage of the total number of Democrats is this?

(b) In each of the four districts won by Democrats, there were 300 Democratic voters and 200 Republicans. Suppose that the same percentage of voters you identified in part (a) switched their votes from Democrat to Republican. Who would now win each of these four districts, and what would the vote totals be?

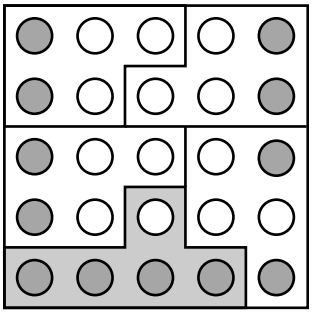


FIGURE 12.12. Measuring partisan fairness

(c) In the remaining district, which was won by Republicans, there were 400 Republican voters and 100 Democrats. What would happen if the same percentage of Democratic voters you identified in part (a) switched their votes? Would the district still be won by Republicans?

(d) Combine your answers to parts (a)–(c) to determine how many of the five districts Republicans would win if they won 52% of the vote statewide.

(e) Does your answer to part (d) seem fair? Does this districting plan treat Democrats and Republicans equally? Why or why not?

You may have noticed a lack of symmetry in your answers to Question 12.9. When Democrats earned 52% of the vote statewide, they won four of the five districts. And when Republicans earned 52% of the vote statewide, Democrats *still* won four of the five districts. This is evidence that the districting plan is biased in favor of Democrats. In fact, if you take a closer look at the plan, you'll see several examples of packing and cracking.

The term *partisan bias* has a precise definition in the context of redistricting, based on the idea that when the statewide vote is split evenly, each party should win the same number of districts.

Definition 12.10. Suppose that, under the assumption of uniform partisan swing, a party wins $x\%$ of the districts when they earn 50% of the statewide vote. The **partisan bias** with respect to this party is equal to $(x - 50)\%$, where a positive result indicates bias in favor of the party and a negative result indicates bias against the party.

Question 12.11.

(a) What is the partisan bias with respect to Democrats in the districting plan from Figure 12.12?

(b) Assuming uniform partisan swing, what is the minimum percentage of the statewide vote that Democrats could earn and still win four of the five districts?

Although partisan bias and related measures can help us identify when a districting plan gives one party an unfair advantage, they do require us to consider hypothetical scenarios or *counterfactuals*, using simplifying assumptions such as uniform partisan swing to complete the required calculations. Courts have not looked favorably on the use of such counterfactuals. In fact, in a case involving a redistricting plan in Texas (*LULAC* v. *Perry*, 2006), U.S. Supreme Court Justice Anthony Kennedy wrote the following in the Court's majority opinion:

> *The existence or degree of asymmetry may in large part depend on conjecture about where possible vote-switchers will reside. Even assuming a court could choose reliably among different models of shifting voter preferences, we are wary of adopting a constitutional standard that invalidates a map based on unfair results that would occur in a hypothetical state of affairs.*

With that in mind, we'll now turn our attention to a new method for identifying partisan gerrymandering—called the *efficiency gap*—that played a prominent role in the *Gill* v. *Whitford* case recently addressed by the U.S. Supreme Court.

The Efficiency Gap

The *efficiency gap*, introduced by Nicholas Stephanopolous and Eric McGhee in 2015 [**49**], captures the idea that gerrymandering results in wasted votes. In a packed district, the winning party often earns significantly more votes than they would need to win the district. These excess votes are essentially wasted. In a cracked district, the losing party may earn a significant number of votes, but not enough to win. These votes are also wasted. In each case, the wasted votes could potentially make a difference in other, more competitive districts; however, because of where they are located—due to the way the district boundaries are drawn—their impact is neutralized. From each party's perspective, the most efficient use of their votes is to win as many districts as possible by the smallest possible margin, and to have hardly any votes left to waste in the districts they lose.

The efficiency gap formalizes this idea by comparing the number of votes wasted by each party across all of the districts. In a perfectly fair districting plan, each party would waste the same number of votes. Therefore, a large difference in the number of votes wasted between the two parties can be viewed as evidence of partisan gerrymandering.

The precise definition of the efficiency gap is as follows:

Definition 12.12. For each district in a districting plan, we consider the following votes to be wasted:

- all of the votes cast by the losing party;

- all of the votes cast by the winning party in excess of the number needed to win the district by a simple majority.

Let w_A and w_B denote the number of wasted votes for parties A and B, respectively, and let v denote the total number of votes cast. The **efficiency gap** is defined to be

$$EG = \frac{w_A - w_B}{v}.$$

Question 12.13.* Suppose the efficiency gap, as defined above, is positive. Which party has the advantage?

Question 12.14.*

(a) Calculate the number of votes wasted by each party in the districting plan from Figure 12.12. (Again assume that each circle represents 100 voters, and remember that it takes 251 votes to win a district by a simple majority.)

(b) Use your calculations from part (a) to determine the efficiency gap.

(c) Which party is favored by the plan, and how is this reflected in your calculation from part (b)?

We noted earlier that an efficiency gap of zero would, at least in theory, represent a perfectly fair and neutral districting plan. It can also be shown that the efficiency gap is always between -0.5 and 0.5. But how big of an efficiency gap is too big? Based on historical analysis, Stephanopolous and McGhee suggest a threshold of 0.08 for state house plans, and the equivalent of two seats for congressional plans. Under some simplifying assumptions, it can be shown that the two-seat threshold is equal to $2/n$, where n is the number of districts. So, for a state with five congressional districts, the threshold would be $2/5 = 0.4$. This actually doesn't rule out a lot; in fact, it's fairly hard to come up with a plan for five districts that has an efficiency gap greater than 0.4. The efficiency gap is more useful in states with a larger number of districts. For example, the 2016 maps for North Carolina and Pennsylvania (which, as noted earlier, are the subjects of recent or pending court cases) both violate the two-seat efficiency gap threshold, with gaps of 20% and 16%, respectively.

Question 12.15. Consider the districting plan shown in Figure 12.13, which you first considered in Question 12.7.

(a) Calculate the efficiency gap of this plan.

(b) For a state with nine congressional districts, Stephanopoulous and McGhee's recommended threshold is 2/9, or approximately 0.22. How does the plan from Figure 12.13 compare to this threshold?

(c) Suppose that two voters in each district changed their votes from Democrat to Republican. What would be the effect of this change on the efficiency gap?

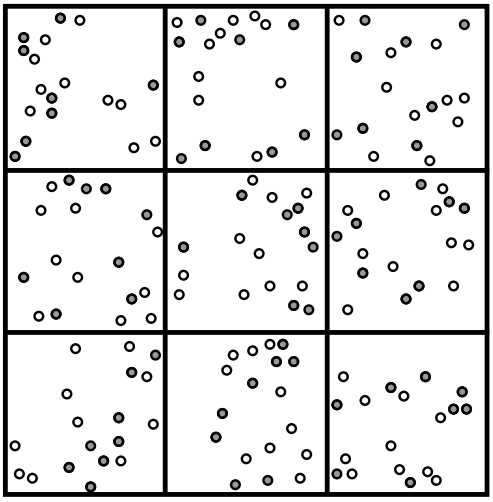


FIGURE 12.13. Gerrymandered or not?

While the efficiency gap is a useful measure that can detect gerrymandering strategies such as packing and cracking with a single number, Question 12.15 illustrates that it is not without limitations. First, a districting plan can have an abnormally high efficiency gap without exhibiting any of the telltale signs of gerrymandering—such as non-compact districts. In the case of the state in Figure 12.13, it is the distribution of the voters themselves, rather than any obvious partisan gerrymandering, that leads to a Democratic sweep of the state's congressional districts. While this outcome may seem unfair, we should also view it in light of our earlier discussions of partisan symmetry. With a reasonable shift in voter preferences and party affiliation—and, importantly, no change in the district boundaries—the tables could easily turn, with the map now favoring Republicans over Democrats and the efficiency gap changing accordingly. So the important question is not only whether a districting plan favors one party over another, but also whether this bias is likely to persist over time and endure natural changes in voter behavior. Since the efficiency gap can be sensitive to small changes, Stephanopolous and McGhee recommend carrying out "sensitivity analysis" when using the efficiency gap to evaluate a plan. Unfortunately, such analysis involves considering hypothetical scenarios—an approach that, as we have discussed, is not viewed favorably by the courts.

Concluding Remarks

Gerrymandering can be a significant barrier to democratic representation in elected bodies such as the U.S. House of Representatives. The problem is exacerbated by the fact that, in most states, it is the state legislatures

themselves that draw the district lines—a role that provides both opportunities and incentives for partisan manipulation. Some states have tackled the issue by appointing independent or bipartisan commissions, and Iowa uses a unique model where an advisory commission proposes a plan to the state legislature for an up-or-down vote. These approaches can be a step in the right direction, but they likely won't completely eliminate gerrymandering or lead to universally accepted solutions.

Complicating all of this is the fact that the law, particularly as it pertains to partisan gerrymandering, is still very much in flux. However, as we've seen, mathematics plays a key role in helping courts to interpret important concepts such as compactness and partisan bias. Judges are relying on the work of mathematicans (and other mathematically inclined scholars in fields such as law and the social sciences) to more precisely define standards that can be used to determine when a districting plan should be upheld or struck down. As we've seen in this chapter, none of the measures for detecting gerrymandering are perfect. They all have limitations, and the nuances of any particular districting plan are probably best understood by considering a variety of different measures. With that said, experts who are well versed in these methods can provide valuable testimony—particularly in response to claims that a gerrymandered plan arose naturally or for legitimate reasons. When a defendant claims that it is not possible to come up with a plan that is more compact or has a lower efficiency gap, you can count on mathematicians—armed with computer simulations and pages of careful analysis—to say, "Sure it is—and here are 500 examples to prove it."

Questions for Further Study

Question 12.16. Revisit the example shown in Figure 12.1, but this time assume that you must divide the state into seven districts. What is the least number of districts that Republicans can win? What is greatest number of districts they can win? What do you think the most fair outcome would be? Give specific plans, with explanation, to justify each of your answers.

Question 12.17. Research each of the following compactness measures, describe how they work, and apply them to the shapes in Figure 12.9.

(a) Polsby-Popper

(b) Schwartzberg

(c) Reock

(d) Convex hull

Question 12.18. The *convexity coefficient* is a compactness measure that assigns a score to a district based on the probability that a line segment drawn between two randomly selected points within the district will remain entirely within the district. In other words, the more lines that cross the district's boundaries, the lower the convexity coefficient will be.

(a) Based on this informal definition, which shapes from Figure 12.9 do you think have the highest convexity coefficients? Which have the lowest convexity coefficients?

(b) Learn more about the convexity coefficient by reading the article "Gerrymandering and convexity" in the *College Mathematics Journal* [**28**], and write a summary of your findings.

(c) If you have experience with probability and statistics, try to calculate the exact convexity coefficient for as many of the shapes from Figure 12.9 as you can. (Hint: For most of the shapes, you'll have to consider several cases. However, some are easier than others.)

Question 12.19. Research the *coastline paradox*, and explain how it is related to certain compactness measures.

Question 12.20. Compare and contrast the idea of partisan symmetry with the neutrality criterion we considered for voting systems.

Question 12.21. Do you think the assumption of uniform partisan swing is reasonable? Why or why not?

Question 12.22. One way to investigate partisan symmetry is to calculate the number of seats that would be won by a party for a variety of possible vote percentages (again using the assumption of uniform partisan swing). The graphs in Figure 12.14 (called *seats-votes curves*) show the results of these calculations for four different districting plans. The x-axis displays the proportion of votes won by the Republican party, while the y-axis displays the number of seats (districts) won by Republicans. So, for example, if the point $(0.4, 0.6)$ is on the graph, this means that when Republicans earn 40% of the vote, they will win 60% of the seats.

(a) For each plan, use the seats-votes curve to decide whether the plan is biased in favor of Republicans, Democrats, or neither. Explain your reasoning.

(b) How do these graphs show evidence of packing and/or cracking? Which features correspond to each of these gerrymandering strategies?

Question 12.23. Research some critiques of the efficiency gap as a measure of partisan fairness, and write a summary of your findings.

Question 12.24. Explain why the efficiency gap is always between -0.5 and 0.5.

Question 12.25. Under the assumptions that the number of votes cast in each district is identical and that the vote is split between exactly two parties, it can be shown that the efficiency gap is equal to $2V - S - \frac{1}{2}$, where V is the proportion of votes received statewide by one of the parties and S is the proportion of seats (districts) won by that party. Use algebra to show

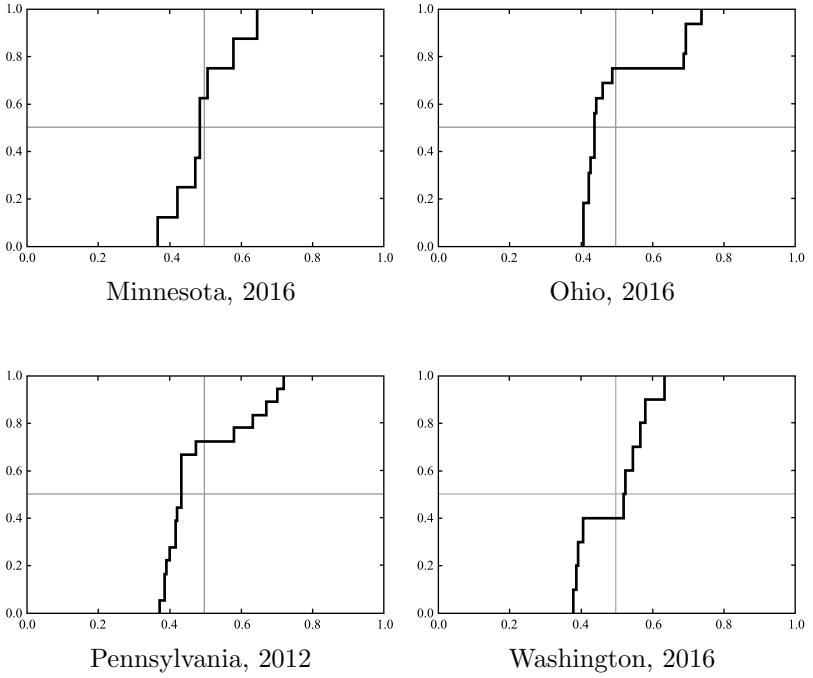


FIGURE 12.14. Seats-votes curves

why this formula holds. Then explain why, to maintain an efficiency gap of zero, every 1% increase in the number of votes received by a party should lead to a 2% increase in the number of seats that party receives.

Question 12.26. How must the vote in a district be split in order for each party to waste the exact same number of votes? Do you think a districting plan in which each district had exactly this split would be fair and/or desirable? Why or why not?

Question 12.27. On February 24, 2018, President Donald Trump sent the following tweet: "Democrat judges have totally redrawn election lines in the great State of Pennsylvania. @FoxNews. This is very unfair to Republicans and to our country as a whole. Must be appealed to the United States Supreme Court ASAP!" Research the history behind this tweet, and explain whether you agree or disagree with Trump's position.

Question 12.28. Research one of the court cases mentioned in this chapter; ideally one that had not been resolved when the book was written. Write a detailed summary of your findings, and explain the impact of the case on the law regarding gerrymandering.

Question 12.29. Look up a recent news article on gerrymandering, and use what you learned in this chapter to either critique the article or respond to the points made in it.

Answers to Starred Questions

12.2. The one district won by Democrats is extremely packed: It doesn't have a single Republican voter! The other Democratic votes are cracked among the remaining four districts, with two Democrats per district—just shy of what would be needed to win.

12.4. (a) Using length-width compactness, shapes (i) and (iii) are tied for being the most compact, and shape (iv) is the least compact.

(b) Using perimeter compactness, shape (ii) is the most compact, and shape (v) is the least compact.

12.5. The ordering of the shapes, from most compact to least compact, is (iii), (ii), (i), (iv), (v).

12.9. (a) If 100 Democrats (about 7.69%) switched their votes, the new statewide totals would be 1300 Republicans (52%) and 1200 Democrats (48%).

(b) 23 Democrats in each of these four districts would change their votes, but Democrats would still have a majority.

(c) 8 Democrats would change their votes, increasing the Republican majority in this already packed district.

(d) Republicans would still win only one of the five districts.

(e) This doesn't seem fair. When Democrats had 52% of the vote statewide, they won four of the five districts. But if Republicans were to earn 52% of the statewide vote, they would only win one of the five districts.

12.13. As it is defined here, a positive efficiency gap would indicate that party A wasted more votes than party B, meaning that party B has the advantage.

12.14. (a) In the four districts that Democrats win, they waste 49 votes, while Republicans waste 200. In the remaining district, Democrats waste 100 votes, while Republicans waste 149.

(b) In total, Democrats waste 296 votes, whereas Republicans waste 949. The efficiency gap is $\frac{949-296}{2500} \approx 0.26$.

(c) The positive efficiency gap from part (b) indicates that the plan favors Democrats.

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The *Mathematics of Voting and Elections: A Hands-On Approach*, Second Edition, is an inquiry-based approach to the mathematics of politics and social choice. The aim of the book is to give readers who might not normally choose to engage with mathematics recreationally the chance to discover some interesting mathematical ideas from within a familiar context, and to see the applicability of mathematics to real-world situations. Through this process, readers should improve their critical thinking and problem solving skills, as well as broaden their views of what mathematics really is and how it can be used in unexpected ways. The book was written specifically for non-mathematical audiences and requires virtually no mathematical prerequisites beyond basic arithmetic. At the same time, the questions included are designed to challenge both mathematical and non-mathematical audiences alike. More than giving the right answers, this book asks the right questions.

The book is fun to read, with examples that are not just thought-provoking, but also entertaining. It is written in a style that is casual without being condescending. But the discovery-based approach of the book also forces readers to play an active role in their learning, which should lead to a sense of ownership of the main ideas in the book. And while the book provides answers to some of the important questions in the field of mathematical voting theory, it also leads readers to discover new questions and ways to approach them.

In addition to making small improvements in all the chapters, this second edition contains several new chapters. Of particular interest might be Chapter 12 which covers a host of topics related to gerrymandering.

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